# Dynamic abstraction

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# Dynamic abstraction

**What is it?**

- particularly simple and unproblematic steps of abstraction, which typically expand the domain
- these steps are iterated a great number of times

**Why do it?**

- the bad company problem remains intractable on the ordinary 'static' approach but disappears on the dynamic approach
- we have a philosophical account of how the simple abstraction steps work
- we get a nice account of indefinite extensibility, which is valuable e.g. for abstractionist set theory

# Russell's paradox

Consider a plural version of Basic Law V:

$$\{xx\} = \{yy\} \leftrightarrow \forall u(u \prec xx \leftrightarrow u \prec yy) \tag{V}$$

Define membership à la Frege:

$$x \in y \leftrightarrow \exists zz(y = \{zz\} \wedge x \prec zz)$$

Let *rr* be all the non-self-membered sets:

$$\forall u(u \prec rr \leftrightarrow u \notin u) \tag{1}$$

Let $r = \{rr\}$. Instantiating $\forall u$ in (1) with respect to $r$ yields a contradiction.

All would be fine if only $r$ wasn't in the range of $\forall u$! The problem is that $\forall u$ was supposed to be unrestricted.

# Predicative vs. impredicative abstraction principles

The background second-order logic can be predicative or impredicative

An abstraction principle

$$\S\alpha = \S\beta \leftrightarrow \Phi(\alpha, \beta) \tag{$\Sigma$}$$

is impredicative if the identity of the new objects (the abstracta) is a matter of how thing stand with the old objects; otherwise, it is predicative.

**Examples**

- Basic Law V and Hume's Principle are impredicative
- Frege's direction principle is predicative

$$d(l_1) = d(l_2) \leftrightarrow l_1 \parallel l_2 \tag{Dir}$$

# Two-sorted abstraction (I)

One way to enforce predicativity is by formulating $(\Sigma)$ in a two-sorted language $\mathcal{L}_1$, which extends a one-sorted base language $\mathcal{L}_0$.

**Advantages of two-sorted abstraction**

- Allows 'entailment preserving' translation from $\mathcal{L}_1$ back into $\mathcal{L}_0$

$$\begin{aligned} d(l_1) =_1 d(l_2) &\quad\mapsto\quad l_1 \parallel l_2 \\ \textsc{Ortho}_1(d(l_1), d(l_2)) &\quad\mapsto\quad \textsc{Ortho}_0(l_1, l_2) \end{aligned}$$

- So satisfies Frege's 'elimination requirement'

  *In our present case, we have to explain the meaning ['Sinn'] of the sentence 'the number which belongs to the concept F is the same as that which belongs to the concept G'; that is to say, we must reproduce the content of this sentence in other terms, avoiding the use of the expression 'the number which belongs to the concept F'. (Grundlagen §62)*

- No bad company problem

# Two-sorted abstraction (II)

What has been achieved?

- One view: a new *façon de parler*, precisely *as if* a second domain has been added
- My view: a new interpretation which adds a second domain of abstracta (Linnebo, 2012)

**Disadvantages of two-sorted abstraction**

- Two-sorted approach is clumsy
- Mathematically very weak

> *Predicative abstraction principles [. . . ] are completely harmless or, if you prefer, useless: they cannot serve as a foundation for mathematics. (Potter, "Abstractionist Class Theory")*

# A better way to enforce predicativity (I)

We can 'merge the sorts' of $\mathcal{L}_1$ to arrive at a one-sorted language $\mathcal{L}_2$.

Although we cannot *translate* from $\mathcal{L}_2$ to $\mathcal{L}_0$, we can provide each formula of $\mathcal{L}_2$ with an *assertibility condition* in $\mathcal{L}_0$

- each line is tagged with '0' or '1' to indicate whether it should be treated as a presentation of a line or of its direction
- each predicate $F$ of $\mathcal{L}_2$ is assigned an assertibility condition $\phi_F$ s.t.

$$F \text{ is assertible of } \langle l_1, t_1 \rangle, \ldots, \langle l_n, t_n \rangle \text{ iff } \phi_F(\langle l_1, t_1 \rangle, \ldots, \langle l_n, t_n \rangle)$$

- e.g. $x_1 = x_2$ is assertible of $\langle l_1, t_1 \rangle, \langle l_2, t_2 \rangle$ iff $\big((t_1 = t_2 = \text{`0'} \wedge l_1 = l_2) \vee (t_1 = t_2 = \text{`1'} \wedge l_1 \parallel l_2)\big)$
- predication of directions must respect parallelism
- in the case of plural BLV, we begin by letting $\mathcal{L}_2$ be singular

# A better way to enforce predicativity (II)

Some freedom in how Caesar-type questions are answered, but . . .

- with (Dir), natural to let each direction be distinct from each line
- with plural BLV, natural to identify $\{xx\}$ with any 'earlier' set with $xx$ as elements

What has been achieved?

- one view: new *façon de parler*, precisely *as if* the domain expands
- my view: a new interpretation with an expanded domain (Linnebo, 2012)

Comparison with two-sorted route to predicativity

- predicativity can also be enforced by assertibility conditions in $\mathcal{L}_0$
- no 'bad company problem'
- one-sorted: domain expansion rather than new domain for a new sort
- still mathematical weak?

# Dynamic abstraction

- with (Dir), no new objects by iterating the abstraction step
- with plural BLV, we get new objects for each iteration of this step

Basic Law V can be 'factored' into a criterion of existence and a criterion of identity.

$$\forall uu\, \exists x\, \textsc{Set}(x, uu) \tag{$V^{\exists}$}$$

$$\textsc{Set}(x, uu) \wedge \textsc{Set}(y, vv) \rightarrow [x = y \leftrightarrow \forall z(z \prec uu \leftrightarrow z \prec vv)] \tag{$V^{=}$}$$

Let's use modal operators to represent what is achieved by the abstraction steps, e.g.

$$\Box \forall xx\, \Diamond \exists y\, \textsc{Set}(y, xx) \tag{$V^{\exists\Diamond}$}$$

**NB!** A plurality is tracked extensionally from 'world' to 'world'.

# A modal representation of a process of individuation

- The accessibility relation $\leq$ is a partial order (i.e. reflexive, transitive, and anti-symmetric).
- $\leq$ is well-founded.
- $\leq$ is directed, i.e.

$$\forall w_1 \forall w_2 \exists w_3 (w_1 \leq w_3 \land w_2 \leq w_3)$$

- *Maximality*: all the objects that can be individuated are individuated.

**NB!** We can now justify directedness by showing how to 'sow together' different extensions resulting from abstraction. Maximality too is fine provided there is a fixed totality of ways of abstracting.

# A modal logic appropriate for the process of individuation

- since $\leq$ is reflexive and transitive, at least S4
- the directedness of $\leq$ ensures the soundness of

$$\Diamond\Box p \rightarrow \Box\Diamond p. \tag{G}$$

- so we adopt S4.2 = S4 + (G)
- since the domains always increase along $\leq$, Converse Barcan holds:

$$\exists\Diamond\phi \rightarrow \Diamond\exists\phi \tag{CBF}$$

'Kicking away the ladder': all the modal assumptions of the formal arguments to follow are available in this theory

# Recovering absolute generality (I)

- $\forall$ and $\exists$ express *relative* or *intraworld* generality
- I claim that $\Box\forall$ and $\Diamond\exists$ express *absolute* or *transworld* generality
- So will call these strings *modalized quantifiers*

**The potentialist translation**: Let $\phi^\Diamond$ be the result of replacing every quantifier in $\phi$ by the corresponding modalized quantifier, e.g.:

$$\forall x \exists y(x = y) \quad \mapsto^\Diamond \quad \Box\forall x\, \Diamond\exists y(x = y)$$

# Recovering absolute generality (II)

**Theorem (Mirroring)**

*Let $\vdash^{\Diamond}$ be provability by $\vdash$, S4.2, and axioms stating that every atomic predicate is rigid, but with no higher-order comprehension. Then we have:*

$$\phi_1, \ldots, \phi_n \vdash \psi \quad \textit{iff} \quad \phi_1^{\Diamond}, \ldots, \phi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}.$$

| | relative generality | absolute generality |
|---|---|---|
| 'for all' | $\forall$ | $\Box\forall$ |
| 'there is' | $\exists$ | $\Diamond\exists$ |

Is this a stable view? Why not transcend $\Box\forall$ as we did $\forall$?

- One option: $\Box$ as schematic generality; $\Diamond$ as procedure for extending.
- Then duality goes, and with it, negative stability (i.e. $\neg\phi^{\Diamond} \to \Box\neg\phi^{\Diamond}$).
- So pressure to translate $\neg\phi \overset{\tau}{\mapsto} \Box\neg\phi^{\tau}$ instead of $\neg\phi \overset{\Diamond}{\mapsto} \neg\phi^{\Diamond}$.
- This validates intuitionistic logic!

# What pluralities are there?

**The easy question**

Any formula $\phi(u)$ *actually* defines a plurality:

$$\exists xx \, \forall u[u \prec xx \leftrightarrow \phi(u)] \qquad \text{(Comp)}$$

**The hard question**

When could there be a plurality that is *necessarily* defined by $\phi^{\Diamond}(u)$?

$$\Diamond \exists xx \, \Box \forall u[u \prec xx \leftrightarrow \phi^{\Diamond}(u)] \qquad (\text{Comp}^{\Diamond})$$

**Answer to the hard question**

- A plurality has the same elements in every possible world.
- Say that $\phi^{\Diamond}(u)$ is *indefinitely extensible* iff it's impossible to exhaust all the $\phi$'s
- So (Comp$^{\Diamond}$) is permissible iff $\phi^{\Diamond}(u)$ is not IE.
- On the non-dual approach, must additionally require ($\tau$ of) decidability:

$$\Box \forall u(\Box \neg \phi^{\tau}(u) \vee \Box \phi^{\tau}(u))$$

# Recovering standard set theory (ZF)

**Theorem**

*Assume*

- *plural logic + the modal logic S4.2 + "maximality"*
- *trans-world extensionality principles for pluralities and sets*
- *potential plural collapse:* $\Box\forall xx\,\Diamond\exists y\,\textsc{Set}(y, xx)$

*Then we can interpret Zermelo set theory minus Infinity and Foundation.*

**Theorem**

*Assume further that every possibility witnessed by the potential universe is witnessed by some possible world:*

$$\phi^{\Diamond} \rightarrow \Diamond\phi \qquad \text{(Refl)}$$

$$\Diamond\forall x(\phi^{\Diamond} \rightarrow \phi) \qquad (\text{Refl}^{+})$$

*(Refl) allows us to interpret Infinity, and (Refl$^{+}$), also Replacement.*

Linnebo, Ø. (2010).
Pluralities and Sets.
*Journal of Philosophy*, 107(3):144–164.

Linnebo, Ø. (2012).
Reference by abstraction.
*Proceedings of the Aristotelian Society*, 112(1pt1):45–71.

Linnebo, Ø. (2013).
The Potential Hierarchy of Sets.
*Review of Symbolic Logic*, 6(2):205–228.

Parsons, C. (1983).
Sets and Modality.
In *Mathematics in Philosophy*, pages 298–341. Cornell University Press, Cornell, NY.

Studd, J. (2014).
Abstraction Reconceived.
forthcoming in the British Journal of Philosophy of Science.