Hilbert's non-formalistic view of mathematics

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Formalism in general

… The various philosophies that go by the name of 'formalism' pursue a claim that the *essence* of mathematics is the manipulation of characters. A list of the characters and allowed rules all but exhausts what there is to say about a given branch of mathematics. According to the formalist, then, mathematics is not, or need not be, about anything, or anything beyond typographical characters and rules for manipulating them.

(Stewart Shapiro: Thinking About Mathematics, Oxford University Press 2000, p. 140)

Two Examples of Formalism

Jean Dieudonné
… mathematics becomes a game, whose pieces are graphical signs that are distinguished from one another by their form.

Haskell B. Curry
The formalist definition of mathemtics is then this: mathematics is the science of formal systems. The propositions of mathematics are the propositions, elementary or metatheoretical, of some formal system or set of systems.

Hilbert's Proof Theory

1. To enumerate all the symbols used in mathematics and logic. …

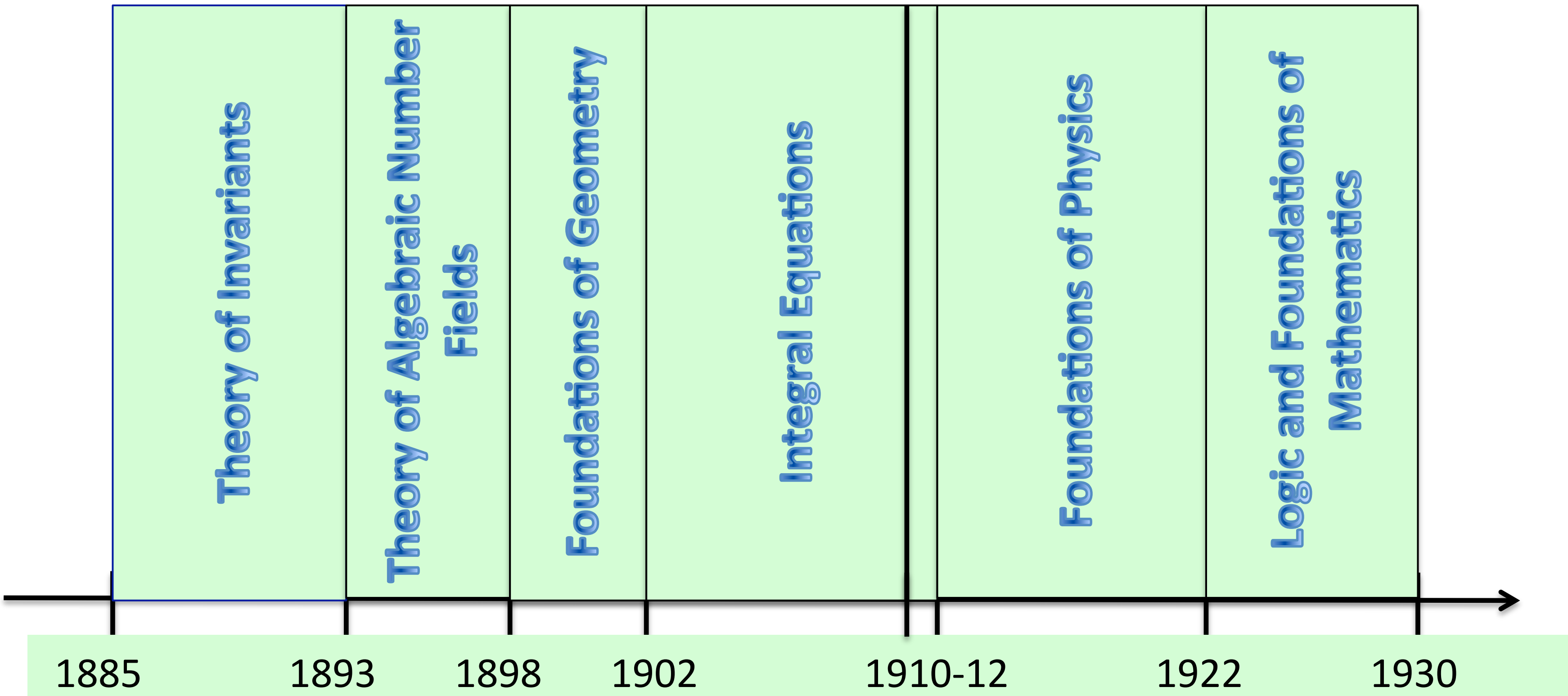
2. To characterize unambiguously all the combinations of these symbols which represent statements classified as "meaningful" in classical mathematics. These combinations are called "formulas".

3. To supply a construction procedure which enables us to construct successively all the formulas which correspond to the "provable" statements of classical mathematics. This procedure, accordingly, is called "proving".

4. To show (in a finitary combinatorial way) that those formulas which correspond to statements of classical mathematics which can be checked by finitary arithmetical methods can be proved (i.e. constructed) by the process described in (3) if and only if the check of the corresponding statement shows it to be true.

(John von Neumann: The formalist foundations of mathematics, P. Benacerraf and Hi Putnam: Philosophy of mathematics, Cambridge UP 1983, p. 63)

Main Periods of Hilbert's Scientific Work



Hilbert's Involvement with Physics

Hilbert's involvement with physical issues spanned most of his active scientific life, and the essence of his mathematical conceptions cannot be understood without reference to that involvement. More importantly, **the famous "axiomatic approach" that came to be identified with Hilbert's mathematical achievements and with his pervasive influence on twentieth-century mathematics is totally misunderstood if it is not seen, in the first place, as connected with his physical interests.**

(Leo Corry: On the origins of Hilbert's sixth problem: physics and the empiricist approach to axiomatization, Proceedings of the International Congress of Mathematicians, Madrid, Spain, 2006 . P. 1699.)

Hilbert's Physics Courses

- 1898 Mechanics
- 1902 Selected Topics in Potential Theory
- 1902/03 Continuum Mechanics – I-II
- 1905/06 Mechanics
- 1906/07 Continuum Mechanics
- 1910/11 Mechanics
- 1911/12 Statistical Mechanics
- 1912 Radiation Theory
- 1912 Mathematical Foundations of Physics
- 1912/13 Molecular Theory of Matter
- 1912/13 Mathematical Foundations of Physics
- 1913 Electron Theory
- 1913/14 Electromagnetic Oscillations
- 1913/14 Analytical Mechanics
- 1914 Statistical Mechanics
- 1915 Structure of Matter (Born's Theory of Crystals)
- 1916/17 Foundations of Physics I-II (General Relativity)
- 1917 Electron Theory
- 1918/19 Space and Time
- 1920 Mechanics
- 1920 Higher Mechanics and the New Theory of Gravitation
- 1921 Einstein's Gravitation Theory. Basic Principles of the Theory of Relativity
- 1922 Statistical Mechanics
- 1922/23 Mathematical Foundations of Quantum Theory
- 1923 Our Conception of Gravitation and Electricity
- 1924 Mechanics and Relativity Theory
- 1926/27 Mathematical Methods of Quantum Theory
- 1930 Mathematical Methods of Modern Physics
- 1931/32 Philosophical Foundations of Modern Natural Science

Geometry: The Science of Space

Geometry is the science dealing with the properties of space. It differs essentially from pure mathematical domains such as the theory of numbers, algebra, or the theory of functions. The results of the latter are obtained through pure thinking ...The situation is completely different in the case of geometry**. I can never penetrate the properties of space by pure reflection**, much the same as I can never recognize the basic laws of mechanics, the law of gravitation or any other physical law in this way. **Space is not a product of my reflections. Rather, it is given to me through the senses**.

(Course notes from Mechanics 1898-9, quoted from Corry,L.,*David Hilbert and the Axiomatization of Physics (1898–1918)*: From *Grundlagen der Geometrie* to *Grundlagen der Physik.* Archimedes: New Studies in the History and Philosophy of Science and Technology 10, Kluwer Academic Publishers, Dordrecht 2004, p.84)

Geometry: A Branch of Physics

The Einsteinian theory of gravitation makes it manifest: geometry is nothing more than a branch of physics; **geometrical truths are in no principled way whatsoever different from physical truths**. Thus, for example, the **Pythagorean theorem** and the Newtonian laws of attraction are essentially related in that they are both governed by the same physical concept, that of the **potential**.

(David Hilbert: *Logic and the Knowledge of Nature. 1930* in William B. Ewald: *From Kant to Hilbert*. Oxford Science Publications.1996. p. 1163)

Geometry: En experimental Science turned into pure mathematics

Geometry also [like mechanics] emerges from the observation of nature, from experience. To this extent**, it is an *experimental science*....But its experimental foundations are so irrefutably and so *generally acknowledged*, they have been confirmed to such a degree, that no further proof of them is deemed necessary.** Moreover, all that is needed is to derive these foundations from a **minimal set of *independent axioms* and thus to construct the whole edifice of geometry by *purely logical means*.** In this way [i.e., by means of the axiomatic treatment] geometry is turned into a *pure mathematical* science. **In mechanics it is also he case that all physicists recognize its most basic facts. But the *arrangement* of the basic concepts is still subject to changes in perception ...and therefore mechanics cannot yet be described today as a *pure mathematical* discipline, at least to the same extent that geometry is.**

(Course notes from Mechanics 1898-9, quoted from Corry,L., *David Hilbert and the Axiomatization of Physics(1898–1918)*:From *Grundlagen der Geometrie* to *Grundlagen der Physik.* Archimedes: New Studies in the History and Philosophy of Science and Technology 10, Kluwer Academic Publishers, Dordrecht 2004, p.90)

Framework of Concepts [Fachwerk von Begriffen]

When we assemble the facts of a definite, more-or-less comprehensive field of knowledge, we soon notice that these facts are capable of being ordered. This ordering always come about with the help of a certain **framework of concepts** in the following way: a concept of this framework corresponds to each individual object of the field of knowledge, and a logical relation between concepts corresponds to every fact within the field of knowledge. **The framework of knowledge is nothing other than the *theory* of the field of knowledge**.

(David Hilbert: *Axiomatic Thought, 1918.* in William B. Ewald: *From Kant to Hilbert*. Oxford Science Publications. 1996. p. 1107-8)

Mathematical Method

The essence of mathematical method consists in the consistent elaboration of the procedures which are unique of formal thinking, that is, the logical reasoning as such. **Logical thinking has at its disposal an immeasurable stock of formal relations; and the task is to find those formal relations which can be adjusted to those relations which exist in reality.**

(Hilbert, D., *Natur und Mathematisches Erkennen: Vorlesungen, gehalten 1919-1920 in Göttingen. Nach der Ausarbeitung von Paul Bernays*. Edited and with an English introduction by David E. Rowe, Birkhäuser, Basel 1992, p. 17)

Mathematics: New Discoveries Before Foundations

The edifice of science is not raised like a dwelling, in which the foundations are first firmly laid and only then one proceeds to construct and to enlarge the rooms. **Science prefers to secure as soon as possible comfortable spaces to wander around and only subsequently, when signs appear here and there that the loose foundations are not able to sustain the expansion of the rooms, it sets about supporting and fortifying them**. This is not a weakness, but rather the right and healthy path of development.

(David Hilbert: Logische Principien des mathematischen Denkens, Lecture notes, summer 1905, notes by M. Born, p. 122)

Two Tasks of Mathematics

On the one hand mathematics must develop the systems of relations and investigate their logical consequences as it is done in the disciplines of pure mathematics. This is the **progressive** task of mathematics.

On the other hand, it consists in giving the experience based theories solid structures and best possible simple foundations. In doing that it is useful to express the assumptions in a clear way and always to differentiate between what is an assumption and what is a logical consequence. ... This is the **regressive** task of mathematics.

(Hilbert, D., *Natur und Mathematisches Erkennen: Vorlesungen, gehalten 1919-1920 in Göttingen. Nach der Ausarbeitung von Paul Bernays*. Edited and with an English introduction by David E. Rowe, Birkhäuser, Basel 1992, p. 17)

Existence in Mathematics

We always talk about **existence relative to a definite system** and this system varies according to which theory we consider. So, in **geometry** we are concerned with points, lines and planes, which are characterized by axioms; in **number theory** we are concerned with the system of integers, which are obtained from one by successively adding ones.

…

[**Ideal elements** occur] when a system is extended through introduction of new elements which with respect to the original system appear to be ideal elements.

(Hilbert, D., *Natur und Mathematisches Erkennen: Vorlesungen, gehalten 1919-1920 in Göttingen. Nach der Ausarbeitung von Paul Bernays*. Edited and with an English introduction by David E. Rowe, Birkhäuser, Basel 1992, p. 90)

Two kinds of ideal elements:

- Das Unwirkliche (bloss Gedachte) dem Wirklichen
- Das Vollkommene dem existierenden Unvollkommenen.

The Origine of Mathematical Problems

Surely the first and oldest problems in every branch of mathematics spring from experience and are suggested by the world of external phenomena. Even the rules of calculation with integers must have been discovered in this fashion in a lower stage of human civilization, just as the child of to-day learns the application of these laws by empirical methods.

(David Hilbert: Mathematical Problems. Bull. Amer. Math. Soc., Vol. 37, No. 4, 2000, p. 409)

The Origine of Mathematical Problems

But, in the further development of a branch of mathematics, **the human mind, encouraged by the success of its solutions, becomes conscious of its independence**. **It evolves from itself alone, often without appreciable influence from without, by means of logical combination, generalization, specialization**, by separating and collecting ideas in fortunate ways, new and fruitful problems, and appears then itself as the real questioner. Thus arose the **problem of prime numbers and the other problems of number theory, Galois's theory of equations, the theory of algebraic invariants, the theory of abelian and automorphic functions**; indeed almost all the nicer questions of modern arithmetic and function theory arise in this way.

(David Hilbert: Mathematical Problems. Bull. Amer. Math. Soc., Vol. 37, No. 4, 2000, p. 409)

The Origine of Mathematical Problems

In the meantime, **while the creative power of pure reason is at work, the outer world again comes into play, forces upon us new questions from actual experience**, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus at the same time advance most successfully the old theories. **And it seems to me that the numerous and surprising analogies and that apparently prearranged harmony which the mathematician so often perceives in the questions, methods and ideas of the various branches of his science, have their origin in this ever-recurring interplay between thought and experience.**

(David Hilbert: Mathematical Problems. Bull. Amer. Math. Soc., Vol. 37, No. 4, 2000, p. 409)

Perceptual Experience and Intuition versus Axiomatic method

In fact, perceptual experience and intuition (*Anschauung*) in the Kantian sense of the term (as Hilbert understood it) are the two main motives of Hilbert's images of elementary geometry. **The axiomatic analysis of scientific theories**, which provides the methodological backbone and the main unifying thread of Hilbert's overall images of science, **was not meant as a substitute for these two main components.** Rather, it embodied Hilbert's preferred way to organically combine and articulate them in the framework of a **regulative 'network of concepts' (*Fachwerk von Begriffe*)** that helps clarify their logical interrelationship.

(Leo Corry: *Axiomatics, Empiricism, and Anschauung in Hilbert's Conception of Geometry: Between Arithmetic and General Reltivity* in Jeremy Gray and José Ferreirós (eds.): *The Architecture of Modern Mathematics: Essays in History and Philosophy*. Oxford University Press 2006, p. 155-156)

Hilbert: Math is Inevitable

The development of mathematics is an inevitably process of pure thinking:

Ebenso wie an diesem speziellen Beispiel zeigt sich überhaupt bei einer näheren Betrachtung der mathematischen Theorien, dass die Entwicklung der Mathematik, abgesehen von dem Einfluss, der von den Bedürfnissen der Geometrie und der Physik ausgeht, schon bei der Art, wie sie durch das reine Denken fortschreitet, im wesentlichen eine zwangsläufige ist.

(Hilbert, D., *Natur und Mathematisches Erkennen: Vorlesungen, gehalten 1919-1920 in Göttingen. Nach der Ausarbeitung von Paul Bernays*. Edited and with an English introduction by David E. Rowe, Birkhäuser, Basel 1992, p. 13)

Coherence in diversity

From the viewpoint of pure mathematical thinking the **development of the methods of infinitesimal calculus and function theory is not at all arbitrary**. Rather the conceptual constructions and inference forms in analysis are indispensible tools for the treatment of problems in number theoretical and algebraic research. So the **exponential function** is of decisive significance for problems pertaining to pure equations (i.e. , equations of the form $x^n=1$), and in questions of advanced number theory it is necessary to use the theory of **elliptic functions and elliptic modular function theory**. Also questions about the distribution of prime numbers require the transcendental tools of function theory (die transzendenten Hilfmittel der Funktionentheorie).

(Hilbert, D., *Natur und Mathematisches Erkennen: Vorlesungen, gehalten 1919-1920 in Göttingen. Nach der Ausarbeitung von Paul Bernays*. Edited and with an English introduction by David E. Rowe, Birkhäuser, Basel 1992, p. 13)

Systematic thought development

The different mathematical disciplines form necessary parts in the construction of a systematic thought development [**Gedankenentwicklung**] which proceeds in essentially pre-determined ways by the compulsion of inner reasons. There is no arbitrariness here. **Mathematics is not like a game by which problems are defined by arbitrary rules**, but it is a conceptual system of inner necessities.

(Hilbert, D., *Natur und Mathematisches Erkennen: Vorlesungen, gehalten 1919-1920 in Göttingen. Nach der Ausarbeitung von Paul Bernays*. Edited and with an English introduction by David E. Rowe, Birkhäuser, Basel 1992, p. 14)

No Absolute Certainty

We have demonstrated that **mathematical conceptualization is not arbitrary**, it is rather constructed systematically from both outer and inner reasons. We looked at the fundamental judgements of mathematical disciplines and found that the **absolute certainty of the axioms [Ausgangssätze] is not a necessary requirement for the successful [erfolgreicher] application of the mathematical method**. And we realized that **mathematical inferences cannot be restricted to ordinary logical method**s and, furthermore, that **they are not free from errors and problems**.

(Hilbert, D., *Natur und Mathematisches Erkennen: Vorlesungen, gehalten 1919-1920 in Göttingen. Nach der Ausarbeitung von Paul Bernays*. Edited and with an English introduction by David E. Rowe, Birkhäuser, Basel 1992, p. 35)

Regulative Employment of the Ideas

It we consider in its whole range the knowledge obtained for us by understanding, we find that what is peculiarly distinctive of reason is its attitude to this body of knowledge, is that **it prescribes and seeks to achieve its *systematisation*, that is, to exhibit the connection of its parts in conformity with a single principle**. This unity of reason always presupposes an idea, namely, that of the form of a whole of knowledge – **a whole which is prior to the determinate knowledge of the parts** and which contains the conditions that determine *a priori* for every part its position and relation to the other parts.
(Kant: Critique of Pure Reason, B673)

Mathematical Rigor

It remains to discuss briefly what general requirements may be justly laid down for the solution of a mathematical problem. I should say first of all, this: that **it shall be possible to establish the correctness of the solution by means of a finite number of steps based upon a finite number of hypotheses which are implied in the statement of the problem and which must always be exactly formulated. This requirement of logical deduction by means of a finite number of processes is simply the requirement of rigor in reasoning.**

(David Hilbert: Mathematical Problems. Bull. Amer. Math. Soc., Vol. 37, No. 4, 2000, p. 409)

A Priori Knowledge

Now I admit that already for **the construction of the theoretical framework certain *a priori* insights are necessary** and that they always underlie the genesis of our knowledge. I also believe that mathematical knowledge in the end rest on a kind of intuitive insight [anschaulicher Einsicht] of this sort …. Thus the most general and fundamental idea of the Kantian epistemology retains its significance: namely the philosophical problem of determining that intuitive, *a priori* outlook and thereby of investigating the condition of the possibility of all conceptual knowledge and of every experience. … **The *a priori* is nothing more and nothing less than a fundamental outlook, or the expression of certain indispensable preconditions of thought and experience**. But we must draw the boundary between what we possess a priori and what requires experience differently that Kant: **Kant greatly overestimated the role and the extent of *the a priori*.**

(David Hilbert: *Logic and the Knowledge of Nature. 1930* in William B. Ewald: *From Kant to Hilbert*. Oxford Science Publications.1996. p. 1161-2)

The *A Priori* is the Finite Attitude

…the Kantian theory of the a priori still contains anthropological dross from which it must be liberated: afterwards only the a priori attitude [Einstellung] is left over which also underlies pure mathematical knowledge: **essentially it is the finite attitude which I have characterized in several works**.

(David Hilbert: *Logic and the Knowledge of Nature. 1930* in William B. Ewald: *From Kant to Hilbert*. Oxford Science Publications.1996. p. 1163)