Potential infinity and intuitionistic logic

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1 Introduction

Hellman and Shapiro [2012], [2013] develop a "regions based" account of a one-dimensional continuum. It follows the Aristotelian theme that continua are not composed of points: each part of the "gunky" continuum has a proper part. "Points" can be defined on the structure, as limits of certain sequences of regions, but there is no sense in which such "points" are in, or are part of, regions. Indeed, "points" just *are* equivalence classes of sets of regions. We show that the "points", so defined, are isomorphic to the real numbers, and thus to the so-called "classical" point-set continuum of Cantor and Dedekind. There is work in progress to provide two- and higher-dimensional, Euclidean and various non-Euclidean regions-based continua.

None of these theories follow Aristotle in another, important aspect. Recall that Aristotle, along with just about every major mathematician and philosopher before the nineteenth century, rejected the very notion of the actual infinite. They argue that the only sensible notion is that of potential infinity (at least for scientific or otherwise non-theological purposes).

In *Physics* 3.6 (206a27-29), Aristotle wrote, "For generally, the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different". As Richard Sorabji [2006, 210, 322-3] puts it, "infinity is an extended finitude".

Aristotle, along with ancient medieval, and early modern mathematicians, recognize the existence of certain *procedures* that can be iterated indefinitely, without limit. Examples are the bisection and the extension of line segments. And they made brilliant use of these, in the method of exhaustion. What they reject are what would be the end results of applying these procedures infinitely often: self-standing points or infinitely long regions. They would also object to thinking of a sequence as itself an actually infinite entity—but it may be too much of an anachronism to put the matter in those terms.

The foregoing regions-based theories make essential use of the actual infinite, much in line with contemporary mathematical practice. We also provide a more Aristotelian account of continuity, sticking for now to one dimension.

The first version is obtained by modifying the theory from Hellman and Shapiro [2012], [2013], to bring it more in line with the rejection of actual infinity. It turns out that analogues of several key theorems have to be added in by hand, as new *axioms*.

Our second theory attempts to capture the “potential” nature of Aristotelian continua by using a modal language.

2 A "semi-Aristotelian" continuum

Our formalism begins with classical, first-order logic with identity supplemented with a standard axiom system for second-order logic (or logic of plural quantification, with an unrestricted comprehension axiom for plurals). The more Aristotelian system developed in subsequent sections does not invoke these higher-order resources.

Axioms of Mereology:

1a. Axioms on $x \leq y$ ("x is part of y"): reflexive, anti-symmetric, transitive.

Certain of our axioms and theorems are conveniently stated in terms of a binary relation called "overlaps": "x *overlaps* y": $x \circ y \Leftrightarrow^{df} \exists z(z \leq x \;\&\; z \leq y)$. And we write $x|y$ for $\neg\exists z[z \leq x \;\&\; z \leq y]$, pronounced "$x$ is discrete from y (and vice versa)".

1b. Axiom on $\leq$ and $\circ$: $x \leq y \leftrightarrow \forall z[z \circ x \rightarrow z \circ y]$.

Theorem 1: *Axioms 1a and 1b imply the Extensionality Principle:*

$x = y \leftrightarrow \forall z[z \circ x \leftrightarrow z \circ y]$.

2. Axiom of Fusion or Whole Comprehension: $\exists u \Phi(u) \rightarrow [\exists x \forall y \{y \circ x \leftrightarrow \exists z(\Phi(z) \;\&\; z \circ y)\}]$, where Φ is a predicate of the second-order language (or language of plurals) lacking free x.

This is the axiom that allows the play with infinity, so to speak. We place no restriction on the number of regions that can be "fused". We take it that the infinities here are "actual", since the axiom entails the existence of a *single* region that is the fusion of all of them.

We write $x+y$ for the mereological sum or fusion of x and y , such that $\forall z[z \leq x+y \leftrightarrow (z \leq x \vee z \leq y)]$, and we use $\sum_{n=0}^{\infty} x_n$ to designate fusions of infinitely many things. Of course, in the Aristotelian theory below, we never take the fusion of infinitely many things.

If $x \circ y$, then we write $x \wedge y$ for the *meet* of x and y. It is the fusion of all regions that overlap x and y. So $\forall z[z \leq x \wedge y \leftrightarrow z \leq x \,\&\, z \leq y]$ (and if x and y have no common part, then $x \wedge y$ is undefined). Here is the first place where we take the fusion of some regions, without bothering to making sure that there are only finitely many such regions.

Similarly, if $\exists z(z \circ x \,\&\, \neg(z \circ y))$, then we let $x - y$ be the sum of all regions z that are part of x but discrete from y (and if there is no such z, then $x - y$ is undefined). So $\forall z(z \leq x - y \longleftrightarrow (z \leq x \,\&\, \neg(z \circ y))$.

We let G be the fusion of *all* regions. It is the entire line, as a single region. This, too, is not acceptable to an Aristotelian.

It is convenient to introduce a geometric primitive, $L(x, y)$, to mean "x is (entirely) to the left of y". The axioms for L specify that it is irreflexive, asymmetric, and transitive. And we define '$R(x, y)$', "x is (entirely) to the right of y", as $L(y, x)$.

Now we can introduce an important geometric relation, *betweenness*: $Betw(x, y, z)$ for "y is (entirely) between x and z":

$$Betw(x, y, z) \Leftrightarrow^{df} [L(x, y) \ \& \ R(z, y)] \vee [R(x, y) \ \& \ L(z, y)]$$

It follows that $Betw(x, y, z) \leftrightarrow Betw(z, y, x)$.

$L(x, y)$ obeys the following axioms:

3a. $L(x, y) \vee R(x, y) \rightarrow x|y$.

3b. $L(x, y) \leftrightarrow \forall z, u[\ z \leq x \ \& \ u \leq y \rightarrow L(z, u)]$.

Now we can define an essential notion, that of a "*connected* part of G". Intuitively, such a part has no gaps. The definition is straightforward:

$$Conn(x) \Leftrightarrow^{df} \forall y, z, u[z, u \leq x \;\&\; Betw(z, y, u) \rightarrow y \leq x]. \qquad \text{(Df } Conn\text{)}$$

("Anything lying between any two parts of x is also a part of x.")

Furthermore, we can define what it means for a connected part of G to be *bounded* (on both sides). Assume $Conn(p)$; then

$$Bounded(p) \Leftrightarrow^{df} \exists x, y[Conn(x) \;\&\; Conn(y) \;\&\; Betw(x, p, y \qquad \text{(Df } Bounded\text{)}$$

(A connected region wholly between two others is bounded.) Once we have established that our space is Archimedean, it will follow that boundedness is also sufficient for "finite in extent".

We call bounded connected regions "*intervals*" and write '$Int(j)$', etc., when needed. However, note that, lacking points, we cannot describe intervals as either "open" or "closed", or "half-open".

Using L, we can impose a condition of dichotomy for discrete intervals:

4. Dichotomy axiom: $\forall i, j[i, j$ are two discrete intervals $\rightarrow (L(i, j) \vee L(j, i))]$.

Now we can prove a linearity condition among intervals:

Theorem 2 *(Linearity)*: *Let x, y, z be any three pairwise discrete intervals; then exactly one of x, y, z is between the other two.*

To guarantee that arbitrarily small intervals exist everywhere along G, we adopt the following "gunkyness" axiom:

5. $\forall x \exists j[Int(j)$ & $j < x]$.

An important relation of two intervals is "adjacency", which is defined as follows:

$$Adj(j,k) \Leftrightarrow^{df} j|k \text{ \& } \nexists m[Betw(j,m,k)].$$

(Df $Adjacent$)

In words, two intervals are adjacent if they are discrete, and no region is between them.

The following equivalence relations on intervals will also prove useful: "*j and k are left-end equivalent*" just in case $\exists p[p \leq j \,\&\, p \leq k \& \nexists q(\{q \leq j \vee q \leq k\} \,\&\, L(q,p)]$. "*Right-end equivalent*" is defined analogously. Intuitively, Left- (Right-) end equivalence means that the intervals "share their left (right) ends, or end-points, in common", but our system does not recognize "ends" or "end-points" as objects entering into mereological relations.

One further geometric primitive is useful both in insuring that G is infinite in extent and in recovering, in effect, the rational numbers as a countable, dense subset of the (arithmetic) continuum, viz. *congruence*, as a binary relation among intervals. Intuitively, $Cong(i,j)$ is intended to mean "the lengths of intervals i and j are equal". Thus, we adopt the usual first-order axioms specifying that *Cong* is an equivalence relation. We will sometimes write this as $|i| = |j|$

We come now to a key axiom, crucial to our characterization of G:

6. Translation axiom: Given any two intervals, i and j, each is congruent both to a unique left-end-equivalent and to a unique right-end-equivalent of the other.

In effect, this guarantees that a given length can be "transported" (more accurately, instantiated) anywhere along G, and that these instances are unique as congruent and either left- or right-end-equivalent to the given length. In particular, we can prove

Lemma 1 *Given any two intervals i and j such that $\neg Cong(i, j)$, either there exists an interval $i' < j$ with $Cong(i, i')$;or there exists i'* with *$j < i'$ with $Cong(i, i')$.*

Theorem 3 *(Trichotomy) For any two intervals, i, j, either $|i| = |j|$ or $|i| < |j|$ or $|i| > |j|$.*

One further axiom on congruence is useful and intuitively intended, viz. that congruence respects nominalistic summation of adjacent intervals:

7. Additivity: Given intervals i, j, i', j' such that $Adj(i, j)$, $Adj(i', j')$, $Cong(i, i')$, $Cong(j, j')$, then $Cong(k, k')$, where $k = i + j$ and $k' = i' + j'$.

We now turn to the matter of the bi-infinitude of G. In fact, our axioms already guarantee this, as we can prove.

Theorem 4 *(Bi-Infinity of G) Let any interval i be given; then there exist exactly two intervals, j, k, such that $Cong(i, j)$ & $Cong(i, k)$ & $Adj(i, j)$ & $Adj(i, k)$ & one of j, k is $left$ of i and the other is $right$ of i.*

The proof of this also involves taking the fusion of some intervals and, here too, there is no reason to think, in advance, that the number of intervals so fused is finite.

Since bi-extension obviously iterates, this already insures that G is "bi-infinite" in the sense of containing, as a part, the fusion of the minimal closure of any interval i under the operation of "*bi-extension*" defined in the theorem. But we can do better and also insure that G is *exhausted* by iterating the process of flanking a given interval by two congruent ones as in Bi-Infinity. This is just the Archimedean property. Toward this end, call an interval l an (*immediate*) *bi-extension* of interval i—$BiExt(l, i)$, or $biext(i) = l$—just in case $l = j + i + k$, where j, i, k behave as in the Bi-Infinity theorem.

Lemma 2 *Let* i and j *be intervals such that* $i < j$; *then* $\neg Cong(i, j)$.

Now we can characterize G. Toward that, let X be any class (or plurality) of intervals such that an arbitrary but fixed interval $i \leq G$ is one of the X and such that if $k = biext(j)$ for j any of the intervals of X, then k is also in X. Call such X a "closure of i under $biext$".

Lemma 3 By axiom 2, there is an individual which is the common part of the fusions of each class X which is a closure of i under $biext$, which we call their *meet* or *the minimal closure* i^* *of* i *under* $biext$. (Since i is stipulated to belong to any such X, the meet is non-null, as required in mereology.)

Here we apply Axiom 2, this time to an *explicitly defined infinite* set (or plurality) of intervals.

By the criterion for identity of mereological objects, the meet i^* of Lemma 3 is unique. We now have a theorem characterizing G as this meet. This is the advertised Archimedean property.

Theorem 5 (*Characterization of* G): *Let* G *be the fusion of the objects in the range of the quantifiers of our axioms; then* $G = i^*$, *the fusion of the minimal closure of i under* $biext$.

Proof sketch: Suppose, for a contradiction, that $G \neq i^*$. So there is a region and, by Axiom 5, an interval that is discrete from i^*. Suppose, without loss of generality, that this interval is Right of i. Recall that i^+ is the positive or right half of i^*. Clearly i^+ is connected; and by our betweenness criterion, it is also "bounded"—it has the parts of i^- on its Left, and, by hypothesis, an interval on its Right. So i^+ is an *interval*. Therefore, by the Translation axiom, there is a unique interval $m \leq i^+$ with the properties (1) m is right-end-equivalent to i^+, and (2) $Cong(m, i)$. But this leads to contradiction. So $G = i^*$.

Finally, we establish that any interval has a unique bisection.

Theorem 6 (*Existence and uniqueness of bisections*): *Given any interval i, there exist intervals j, k such that* $j < i$ & $k < i$ & $j|k$ & $j + k = i$ & $Cong(j, k)$; *and* j, k *are unique with these properties.*

Proof sketch: For any interval j, let $2j$ be the fusion of j with an interval that is congruent with j and adjacent to j, say on its Right. One can show that for any interval i there is an interval j such that $|2j| \leq |i|$. The left bisect of i is the fusion of all intervals j such that j is left-end-equivalent to i and $|2j| \leq |i|$. Notice that here, too, we take the fusion of some intervals without knowing (or caring) whether they are infinite in number.

We can now *define* an exact locus or "point" as a "Cauchy sequence" of intervals. Taking the fusion of the members of such a sequence gives us an interval that can play the role of a given real number. But, of course, these "Cauchy sequences" of intervals are themselves infinite and so here, too, we take the fusion of an infinite set of regions. We can show how to embed the real numbers into G, given an arbitrary interval i to serve as a unit.

There is a sort of converse to this as well. If we start with the standard, Cantor-Dedekind continuum, we can produce a model of the foregoing theory. The "regions" are the regular, open sets, i.e., those sets of real numbers that are identical to the interior of their closures. The regular closed sets—those that are the closures of their interiors—constitute another model of the above theory.

3 Basic Aristotelian theories

As noted, the above treatment invokes infinity in two related ways, neither of which is acceptable to an Aristotelian. First, there are regions, such as the entire Gunky line G, that are infinitely long, so to speak. A bit more formally, G is infinitely long relative to any interval. The second and more substantial difference is that in the proofs of some of our theorems, we take fusions of infinite sets (or pluralities or properties) of regions. For an Aristotelian, there is no conceptual problem with there being a *potential* infinity of regions. Ancient geometry establishes that there are certain procedures, like taking bisections and biextensions, and these procedures can be applied over and over, without limit. But, it seems to us, one cannot take the fusion of infinitely many such regions—to produce a *single* region—unless we take the infinity to be "actual".

It turns out that in the Aristotelian theory, developed below, analogues of several of the key *theorems* above have to be added in by hand, as new *axioms*. This at least suggests the fruitfulness of the notion of infinity.

Recall that our play with infinity is sanctioned by Axiom 2, the principle of fusions or whole comprehension:

$$\exists u \Phi(u) \to [\exists x \forall y \{y \circ x \leftrightarrow \exists z (\Phi(z) \ \& \ z \circ y)\}],$$

where Φ is a predicate of the second-order language (or language of plurals) lacking free x.

One natural Aristotelian "fix" to our system would be to replace the above Axiom 2, of unrestricted fusions, with an axiom asserting the existence of fusions of finite sets. For this, of course, it is sufficient to state that any two regions have a fusion:

$$\text{(FINFUS)} \qquad \forall u \forall v \exists x \forall y [y \circ x \leftrightarrow (y \circ u) \vee (y \circ v)].$$

In fact, it will be convenient for us to adopt an alternative but equivalent fix, which will facilitate comparison with both the more standard theories and the modal approach to be developed in the next section. The first such principle states the existence of all "singleton pluralities" and the second states that any object may be added to any plurality to yield a plurality:

$$\forall x \exists xx \forall y (y \prec xx \leftrightarrow y = x) \tag{FIN-P1}$$

$$\forall xx \forall y \exists zz \forall w (w \prec zz \leftrightarrow w \prec xx \vee w = y) \tag{FIN-P2}$$

The fusion principle is

$$\forall xx \exists y (y = \sum xx) \tag{P-FUS}$$

where $y = \sum xx$ is defined as follows:

$$\forall z (z \circ y \leftrightarrow \exists x (x \prec xx \,\&\, z \circ x)).$$

Let A^0 be the theory that results from our semi-Aristotelian theory by removing the axiom for unrestricted fusion and adding FIN-P1, FIN-P2, and P-FUS (or, alternately, adding FINFUS), as our principles for finite fusions.

Unfortunately, A^0 is far too weak. Presumably, an Aristotelian would want to assert the existence of the meet of overlapping intervals, as well as the existence bisections, biextensions and appropriate differences. None of these can be proved in A^0.

Indeed, let X be the set of all open intervals of real numbers in the form $(a, b+\pi)$, where a and b are rational numbers. Define the fusion of a subset x of X to be the union of x, and let Y be the set of all fusions of finite subsets of X. Say that a region a is "Left" of a region b if every member if a is less than every member of b, and let "congruent" have its usual definition, on intervals. It is straightforward to check that Y is a model of A^0—it satisfies our three axioms governing finite fusions and all of the axioms of the above semi-Aristotelian theory except, of course, Axiom 2, the principle of unrestricted fusions.

However, Y does not satisfy differences, bisections or biextensions. For example, the intervals $(0, \pi)$ and $(0, 1+\pi)$ are both in Y, but neither $(0, \pi/2)$, nor $(\pi, 2\pi)$, nor $(\pi, 1+\pi)$ are. So $(0, \pi)$ has neither a bisection nor a right biextension, and the two intervals do not have a difference. Notice, incidentally, that if i and j are any intervals in Y, then i is *not* adjacent to j. That is, no two intervals in Y are adjacent to each other.

The following is perhaps interesting:

> Lemma: If follows from A^0 that if there is at least one pair of adjacent intervals, then *every* interval has a left and right biextension (as in Theorem 4 of the above semi-Aristotelian theory).

The above set Y does have meets for any pair of overlapping regions. To show that A^0 does not guarantee the existence of meets, let Z_0 be the set of all intervals (a, b), where a and b are rational numbers. Let Z_1 be Z_0 together with two more regions, $(0, 2) \cup (3, 5) \cup (6, 8) \cup (9, 11) \cup \ldots$, and $(1, 3) \cup (4, 6) \cup (7, 9) \cup (10, 12) \cup \ldots$ Finally, let Z_2 be the set of fusions of all finite subsets of Z_1, where, as in the topological models of the semi-Aristotelian system, the fusion of a set is the interior of the closure of the union of the set. Notice that Z_2 satisfies the axioms of A^0. The two added regions overlap but they have no meet.

Let A^1 be the result of adding to A^0 an axiom of *differences*:

$$\text{If } \exists z(z \circ x \,\&\, \neg(z \circ y)) \text{ then } \exists w \forall z(z \leq w \longleftrightarrow (z \leq x \,\&\, \neg(x \circ y)).$$

Again, if $\exists z(z \circ x \,\&\, \neg(z \circ y))$ then let $x - y$ be the resulting difference.

Theorem A1: A^1 implies the existence of biextensions of intervals and the meet of any overlapping regions.

We still do not have bisections. To see this, let X be the set of all open intervals (a, b) where a and b are both rational numbers of the form $c/3^d$ where c is an integer and d is a natural number. Clearly, the translation of any member of X is itself a member of X. Let Y be the set of all finite fusions of members of X (where the fusion of a set is the interior of the closure of its union). It is straightforward that Y satisfies A^1. But the interval $(0, 1)$ has no bisect, since .5 is not of the form $c/3^d$.

It would be straightforward to add an axiom of bisections:

For any interval i, there exist intervals $j < i$ and $k < i$ such that j and k are congruent, discrete, and $j + k = i$.

However, we still would not have trisections. So we add a principle asserting that, for each natural number n, there are n-sections:

> (n-SECT) Given any interval i, there exist intervals $j_1, \ldots, j_n$, all congruent to each other, all part of i, and such that for each $m, m' \leq n$,if $m \neq m'$, then j_m is discrete from $j_{m'}$, and $i = j_1 + \cdots + j_n$.

One could just take this as an axiom scheme, one instance for each natural number n. It would perhaps be better to add terminology for the natural numbers to the system, with appropriate axioms. In the Aristotelian spirit, the natural numbers would be thought of as a potential infinity. Let A^2 be the resulting theory, A^1 plus (n-SECT).

We can, however, avoid explicit talk about natural numbers by exploiting an idea that is anyway congenial to Aristotelians, namely that every positive natural number is "instantiated" by a (finite) plurality of objects. The above axioms FIN-P1 and FIN-P2, together with some of the existence axioms above, ensure that every positive natural number is instantiated by some plurality of regions. That is, for each natural number n, there is are some objects such that there are exactly n of them. Moreover, according to Aristotle *any* plurality is a finite plurality.

So our axiom can be expressed as follows:

(n-SECT*) Given any interval i and any objects pp, there exist some intervals jj such that

(i) jj are equinumerous with pp,

(ii) every one of the jj is a part of i,

(iii) any two distinct members of jj are discrete,

(iv) any two members of jj are congruent to each other, and

(v) $i = \sum jj$

Let A^{2^*} be the A^1 plus (n-SECT*).

An Archimedean property does not follow from A^2, nor from A^{2^*}. To see this, just let R be any non-Archimedean ordered field, and let Y be the set of fusions of finite intervals from R (where the fusion of a set is the interior of the closure of its union).

There is more than one Archimedean property that can be formulated here. Aristotle's own definition is this: "for by continual addition to a finite magnitude, I must arrive at a magnitude that exceeds any assigned limit" (*Physics*, 8, 10, 266b3). In Euclid's *Elements* (Book V, Definition 4), the Archimedean principle appears as a definition (or as a consequence of a definition): "Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another". One Archimedean principle is a restriction of the Aristotle/Euclid principle to intervals.

If we assume the natural numbers, it is straightforward to formulate this in our system. If i is an interval and n any natural number greater than zero, define the n-fold biextension of i, written ni, by recursion, as follows:

$1i = i$

$(n+1)i$ is the fusion of ni with an interval congruent with i and adjacent on its left and an interval congruent with i adjacent on its right.

Our principle of bi-extensions guarantees that these intervals exist. Our first Archimedean principle is this:

(ARCH1) Let i and j be intervals. Then there is a natural number n such that j is congruent to a part of ni.

It follows from Theorem 5 above that our semi-Aristotelian theory satisfies (ARCH1).

Following the above, we can express a version of this Archimedean principle in the plural language, eschewing any reference to natural numbers. We define

> if xx is a one-member plurality, then $xx \cdot i = i$
>
> for any xx and any yy, if yy is the result of adding a single object to xx, then $yy \cdot i$ is the fusion of $xx \cdot i$ with an interval congruent with i and adjacent on its left and an interval congruent with i adjacent on its right.

Following our assumption that all pluralities are finite, our axioms guarantee that all of these intervals exist. The relevant Archimedean principle is this:

> (ARCH1*) Let i and j be intervals. Then there are some things xx such that j is congruent to a part of $xx \cdot i$.

The above, semi-Aristotelian theory does prove (ARCH1), and it proves a version of (ARCH1*) where the plural variables restricted to finite pluralities.

In the present context, (ARCH1) and (ARCH1*) each rules out *intervals* that are infinitesimal, or infinite, relative to each other, but neither of them rules out *regions* that are infinitely long, relative to intervals. Moreover, neither theory rules out the existence of intervals that are "infinitely far apart" from each other, relative to a given interval. Of course, Theorem 5 above rules out the latter, in our original "semi-Aristotelian" theory.

The following are stronger Archimedean principles, stating that if we start with an interval, and keep bi-extending it, we eventually include any given *region*:

> (ARCH2) Let i be an interval and let k be any region. Then there is a natural number n such that $k < ni$.
>
> (ARCH2*) Let i be an interval and let k be any region. Then there are some things xx such that $k < xx \cdot i$.

Of course, the semi-Aristotelian theory refutes (ARCH2) and (ARCH2*), since it has infinitely large regions, such as the entire space G. However, we think that an Aristotelian should endorse these Archimedean principles.

Let A^3 be the theory obtained from A^2 by adding (ARCH2); and let A^{3^*} be the theory obtained from A^{2^*} by adding (ARCH2*).

These, finally, are our first Aristotelian theories. It is, we think, interesting that we seem to have to add several rather different axioms in order to recover what an Aristotelian takes to be the legitimate consequences of the rejected axiom of unrestricted fusions.

One model of A^3 and A^{3^*} consists of all fusions of finite sets of open intervals in the real numbers (where the fusion of a set is the interior of the closure of its union). That is, in a sense, the intended model of each theory. Another model consists of the regular open sets of real numbers that have finite measure. Also the regular closed sets of finite measure. Those are analogues of the topological models above.

Another model of A^3 and of A^{3^*} consists of all fusions of finite sets of open intervals in the rational or algebraic numbers. So we have not ruled out countable models. In light of the axiomatization, this is to be expected. Every axiom of A^3 is first-order, except perhaps (ARCH3), which makes direct reference to the natural numbers. So one natural background for A^3 is ω-logic, a formalism with a variable sort (or something equivalent) for the natural numbers, assumed to be standard.

Some of the axioms of A^{3^*} are, of course, formulated in the language of plurals. But these are intended to be restricted to finite pluralities. So a natural background framework for A^{3^*} is a plural analogue of so-called *weak second-order logic*, where the plural quantifiers are restricted to finite pluralities.

The (downward) Löwenheim-Skolem theorem holds for both ω-logic and this variant on weak second-order logic (see Shapiro [1991, Chapter 9]). So we have not adopted a formalism with the resources to rule out countable models.

A fortiori, we have not ruled out models that are, in a sense, Dedekind. incomplete (cf. the Appendix to Hellman and Shapiro [2013]). Indeed, we do not know how to even *state* a principle of Dedekind. completeness in an Aristotelian setting. We'd have to be able to talk about, and have quantifiers ranging over, infinite sets of regions (or points). The same goes for stating a principle of Cauchy completeness, which requires formulating, and quantifying over, infinite sequences of regions, or intervals. In the development of a punctiform "superstructure" below, we take some steps in that direction, in an attempt to take the notion of a potentially infinite sequence seriously. However, we think that this goes beyond Aristotelian resources.

4 Going modal

The "gunkiness" Axiom 5 is that every region has an interval as a proper part. This guarantees that every model of A^3 and A^{3^*} is itself infinite, in the sense that it has infinitely many regions (given the axioms of mereology). So we have not completely avoided infinity—nor do we want to. One can think of the intended domain, informally, as a "merely potential infinity". It is, however, not clear what this amounts to. This section is an attempt to make sense of Aristotle's claim, cited above, that "the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different" (*Physics* 3.6, 206a27-29). Or, as Sorabji [2006, 210, 322-3] puts it, "infinity is an extended finitude". It is clear what an Aristotelian does *not* have: actually infinite totalities. But what does she have instead? Just what *is* a merely potential infinity?

There is a closely related matter. It is generally agreed that Euclid's *Elements* captures at least the spirit of geometry during Plato's and Aristotle's period. Most of the language in the *Elements* is dynamic, talking about what a (presumably idealized) geometer can *do*. For example, the first Postulate is "To draw a straight line from any point to any point", and the second is "To produce a finite straight line continuously in a straight line". Plato was critical of the geometers of his day, arguing that the dynamic language is inconsistent with the nature of the subject matter of geometry:

> [The] science [of geometry] is in direct contradiction with the language employed by its adepts . . . Their language is most ludicrous, . . . for they speak as if they were doing something and as if all their words were directed toward action . . . [They talk] of squaring and applying and adding and the like . . . whereas in fact the real object of the entire subject is . . . knowledge . . . of what eternally exists, not of anything that comes to be this or that at some time and ceases to be. (*Republic*, VII)

Of course, Aristotle rejects this Platonism, and, we think, the dynamic language employed in geometry better reflects his views. The matter of infinity is tied to this. For Aristotle, we never have infinite collections. Instead, we have *procedures* that can be iterated indefinitely, and we speak about what those procedures *could* produce. In holding that the procedures in question can be iterated indefinitely, Aristotle again follows geometric practice, this time in opposition to his other major opponents, the atomists (see Miller [1982]), who postulate a limit to, say, bisection.

When it comes to the infinite, views like Aristotle's were standard throughout the medieval and early modern period, well into the nineteenth century. The greatest mathematical minds insisted that only the potentially infinite makes sense. Leibniz, for example, wrote:

> It could . . . well be argued that, since among any ten terms there is a last number, which is also the greatest of those numbers, it follows that among all numbers there is a last number, which is also the greatest of all numbers. But I think that such a number implies a contradiction . . . When it is said that there are infinitely many terms, it is not being said that there is some specific number of them, but that there are more than any specific number. (Letter to Bernoulli, Leibniz [1863, III 566], translated in Levey [1998, 76-77, 87])

... we conclude ... that there is no infinite multitude, from which it will follow that there is not an infinity of things, either. Or [rather] it must be said that an infinity of things is not one whole, or that there is no aggregate of them. (Leibniz [1980, 6.3, 503], translated in Levey [1998, 86])

Yet M. Descartes and his followers, in making the world out to be indefinite so that we cannot conceive of any end to it, have said that matter has no limits. They have some reason for replacing the term “infinite” by “indefinite”, for there is never an infinite whole in the world, though there are always wholes greater than others ad infinitum. As I have shown elsewhere, the universe cannot be considered to be a whole. (Leibniz [1996, 151])

For Leibniz, as for Aristotle, as for a host of others, the infinite just *is* the limitlessness of certain processes; no actual infinities exist. The only intelligible notion of infinity is that of potential infinity—the transcendence of any (finite) limit.

For at least the cases of interest here—regions, natural numbers, real numbers, and the like—Cantor argued for the exact opposite of this, claiming that the potentially infinite is dubious, unless it is somehow backed by an actual infinity:

> I cannot ascribe any being to the indefinite, the variable, the improper infinite in whatever form they appear, because they are nothing but either relational concepts or merely subjective representations or intuitions (imaginationes), but never adequate ideas (Cantor, [1883, 205, note 3]).

> . . . every potential infinite, if it is to be applicable in a rigorous mathematical way, presupposes an actual infinite (Cantor, [1887, pp. 410–411]).

We think it safe to say that this Cantorian orientation is now dominant in the relevant intellectual communities, especially concerning the mathematical domains mentioned above, with various constructivists as notable exceptions. Incidentally, Cantor was not completely consistent here, as he sometimes regarded the entire universe as only potential.

In the notes on a discussion between Tarski, Quine, and Carnap, on January 31, 1941, Carnap says:

> A position is an ordering possibility for a thing. I do not have the intuitive [instinctive] rejection of the concept of possibility as Tarski and Quine do. To me the possibility of always proceeding seems the foundation of number theory. Thus, potential but not actual infinity
>
> (Tarski and Quine say: they do not understand this distinction).

A remarkable turn of events.

4.1 The modality

The task at hand is to formalize the dynamic orientation, and the concomitant notion of potentiality, by using a modal language. In contemporary jargon, we will formulate a principle that each "possible world" has access to another one, which contains all the regions of the first world and possibly some more, such as n-sections, biextensions, and translations of the regions in the first world. But we can assume that each such "world" is finite. The total space of "worlds" is, of course, infinite, but we think of possible worlds as only a manner-of-speaking. The theory itself is formulated in the modal language, with the modal operators primitive.

The present "potentialist" view takes its inspiration from Linnebo [2013], which develops a modal explication of the Cantorian notion that the universe of set theory is itself potential. The present situation is a whole lot simpler, since every world here is (or can be taken to be) finite. We have *no* actual infinities, or, in other words, in the present context, nothing is even (actually) transfinite. Also, here we do not have to worry about higher-order resources, at least on the basic Aristotelian theory.

We must first say something about the modality that we invoke, which motivates a specific modal logic. Then we show how the desired Aristotelian mathematical reasoning can be carried out in that framework. We begin with the former task here. As indicated already, when explaining the modality, it will often be useful to indulge in talk about possible worlds. But, again, this is for heuristic purposes only. The modal logic that our heuristic reasoning motivates will be rock bottom, not explained or defined in terms of anything else.

A notable feature of the modal reasoning we will employ is that the domains of the possible worlds grow along the accessibility relation. So we assume:

$$w_1 \leq w_2 \rightarrow D(w_1) \subseteq D(w_2),$$

where '$w_1 \leq w_2$' says that w_2 is accessible from w_1, and for each world w, $D(w)$ is the domain of w.

As is well known, the conditional entails that the converse Barcan formula will be validated, that is:

$$\exists x \Diamond \phi(x) \rightarrow \Diamond \exists x \phi(x) \tag{CBF}$$

This alone makes it doubtful that the modality in question can be "ordinary" metaphysical modality—whatever exactly that is. For it is widely held that there are objects whose existence is metaphysically contingent (Williamson [2013] to the contrary). But, of course, here our variables are restricted to regions, and perhaps the existence of those is not contingent.

4.1.1 Modal Logic

Returning to the formal development, our next pressing issue concerns the right *logic* for the modality we use to explicate potential infinity. Again, it will be useful to indulge in talk about possible worlds, writing the associated accessibility relation as $\leq$. Recall that $w \leq w'$ means that we can get from w to w' by generating more regions. This motivates the following principle:

> **Partial ordering**: The accessibility relation $\leq$ is a partial order. That is, it is reflexive, transitive, and anti-symmetric.

At any given stage in the process of constructing regions, we will generally have a choice which regions to generate. For example, given two intervals that don't yet have bisections, we can choose to bisect one or the other of them, or perhaps to bisect both simultaneously. Assume we are at a world w_0 where we can choose to generate regions so as to arrive at either w_1 or w_2. It makes sense to require that the licence to generate a region is never revoked as our domain expands. The option can always be exercised at a later stage. This corresponds to the requirement that the two worlds w_1 and w_2 can be extended to a common world w_3. This property of a partial order is called *directedness* and formalized as follows:

$$\forall w_1 \forall w_2 \exists w_3 (w_1 \leq w_3 \,\&\, w_2 \leq w_3)$$

We therefore adopt the following principle.

> **Directedness**: The accessibility relation $\leq$ is directed.

This principle ensures that, whenever we have a choice of regions to generate, the order in which we choose to proceed is irrelevant. Whichever region(s) we choose to generate first, the other(s) can always be generated later. Unless $\leq$ was directed, our choice whether to extend the ontology of w_0 to that of w_1 or that of w_2 might have an enduring effect.

The mentioned properties of the accessibility relation $\leq$ allow us to identify a modal logic appropriate for studying the generation of regions. Since $\leq$ is reflexive and transitive, the modal logic S4 will be sound with respect our intended system of possible worlds. As is well known, the directedness of $\leq$ ensures the soundness of the following principle as well:

$$\Diamond\Box p \rightarrow \Box\Diamond p.$$

The modal propositional logic that results from adding this principle to a complete axiomatization of S4 is known as S4.2.

As already discussed, we also have the Converse Barcan Formula, which means that S4.2 can be combined with an ordinary theory of quantification with no need for any complications such as a free logic or an existence predicate.

4.1.2 Bridging actualist and potentialist theories

We now turn to the second task identified above, namely the mathematical one. As observed, the existence claims of our Aristotelian theories A^{3} and A^{3^*} are to be replaced by corresponding claims about possible existence—about what can be constructed. How will reasoning from these modal assumptions compare with the non-modal reasoning of the previous section? Fortunately, this question admits of a systematic answer, with no need for *ad hoc* maneuvering: there are bridging principles that relate "potentialist" claims with the corresponding "actualist" ones.

Two different kinds of generalization are available in the modal framework. First, there are the generalizations expressed by the ordinary quantifiers $\forall$ and $\exists$. Since the variables range just over the ontology of the relevant world (so to speak), this is an *intraworld* form of generality. That is, a sentence in the form $\forall x \phi$, for example, is true at a world w just in case ϕ holds of all objects in $D(w)$, the domain of w.

But there is also another, *transworld* form of generality available, expressed by the complex strings $\Box\forall$ and $\Diamond\exists$. These strings have the effect of generalizing not just over all entities at the relevant world, but over all entities at all worlds, or at least all accessible worlds. This idea will receive a precise statement in a "mirroring" theorem that we will state shortly. Loosely speaking, this theorem says that, under some plausible assumptions, the strings $\Box\forall$ and $\Diamond\exists$ behave logically just like quantifiers ranging over all entities at all (future) worlds.

Because of the Mirroring Theorem, we refer to these strings as *modalized quantifiers*, although they are strictly speaking composites of a modal operator and a quantifier proper.

Some definitions. First, given a non-modal formula ϕ of L, its *potentialist translation* $\phi^{\Diamond}$ is the formula that results from replacing each ordinary quantifier in ϕ with the corresponding modalized quantifier. That is, '$\forall x$' is replaced by '$\Box\forall x$'; and '$\exists x$' is replaced by '$\Diamond\exists x$'. Say that a formula is *fully modalized* just in case all of its quantifiers are modalized.

Second, we say that a formula ϕ is *stable* if the necessitations of the universal closures of the following two conditionals hold:

$$\phi \rightarrow \Box\phi$$
$$\neg\phi \rightarrow \Box\neg\phi$$

Intuitively, a formula is stable just in case it never "changes its mind", in the sense that, if the formula is true (or false) of certain objects at some world, it will remain true (or false) of these objects at all "later" worlds as well.

Let $\vdash$ be the relation of *classical* deducibility in a language L. Then let $\vdash^{\Diamond}$be deducibility in the classical modal language corresponding to L, by $\vdash$, S4.2, and the stability axioms for all atomic predicates of L. We are now ready to state the mentioned theorem.

For any formulas $\phi_1, \ldots, \phi_n, \psi$ of L, we have:

$$\phi_1, \ldots, \phi_n \vdash \psi \quad \text{iff} \quad \phi_1^{\Diamond}, \ldots, \phi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}.$$

See Linnebo [2013] for a proof.The Mirroring Theorem tells us that, if we are interested in logical relations between fully modalized formulas in a modal theory that includes S4.2 and the stability axioms, we may delete all the modal operators and proceed by the ordinary non-modal logic underlying $\vdash$. Thus, under the assumptions in question, the composite expressions $\Box\forall$ and $\Diamond\exists$ behave logically just like ordinary quantifiers, except that they generalize across all (accessible) possible worlds rather than a single world. This provides the desired bridge between actualist and potentialist theories.

4.2 Mirroring for intuitionists

As noted, Linnebo's lovely mirroring theorem presupposes that the background logic is classical. The purpose here is to extend that, for, some argue, only intuitionistic logic is appropriate for theorizing about the potentially infinite.

It is not all that clear what one should use for a framework to develop an intuitionistic modal logic. A decent choice is in the dissertation of an Alex K. Simpson. The basic idea is to work in a more or less standard possible-worlds-framework for the modal operators. We define the notion of frame and interpretation, giving the usual clauses for the connectives and quantifiers (i.e., *not* the ones in the usual Kripke framework for intuitionism).

So we say, for example, that a formula $\forall x \Phi(x)$ is true at a world w just in case $\Phi(a)$ is true in w for all objects a in the domain of w. This, of course, is different from the clause for the universal quantifier in the Kripke semantics for intuitionistic logic (which requires Φ to be true for all objects in all worlds accessible from w).

We get an intuitionistic modal logic when we use intuitionistic logic in the meta-theory. So we do just that below. For example, the clause for negation is this: $\neg\Phi$ holds at a world just in case it is not the case that Φ holds at that world. We insist that this "it is not the case that" is an intuitionistic negation.

Alex Simpson gives a nice natural deduction framework for the meta-theoretic reasoning, but I think we can leave things at a less formal level here. Officially, the derivations themselves mention worlds. One rule, for example, is that if a formula Φ holds at a world w, and if a world w' sees w, then $\Diamond\Phi$ holds at w'. When convenient, I'll suppress mention of the world, assuming (when I do) that it is the same world throughout.

As with Øystein's framework, the accessibility relation is reflexive, transitive, and directed. Directedness is that if $w \leq w_1$ and $w \leq w_2$, then there is a world w_3 such that $w_1 \leq w_3$ and $w_2 \leq w_3$.

The modal principle behind this is:

(G) $\qquad \Diamond\Box\Phi \rightarrow \Box\Diamond\Phi.$

Øystein's motivation for S4.2 carries over.

Recall that a formula Φ is said to be *stable* in a given theory if both

$$\Phi \rightarrow \Box\Phi \text{ and}$$

$$\neg\Phi \rightarrow \Box\neg\Phi$$

are both provable in that theory.

The complex strings $\Box\forall x$ and $\Diamond\exists x$ are called *modalized quantifiers*. A formula is called *fully modalized* if all of the quantifiers in it are modalized. And if Φ is a formula in a non-modal language, let $\Phi^{\Diamond}$ the be the result of replacing each quantifier in Φ with its modalized counterpart.

Let the stability axioms for a modal language be statements that each atomic predication is stable.

Øystein proves the following lemma (5.3):

Let Φ be a fully modalized formula in a modal language. Then S4.2 and the stability axioms for that language prove that $\Diamond\Phi$, Φ, and $\Box\Phi$ are equivalent.

Øystein's proof makes essential use of excluded middle (this is not a criticism). The lemma fails, as stated, in the intuitionistic framework. To see this, consider a frame with two worlds, w and w' , where each world sees itself, $w \leq w'$, but not conversely. Let A be any atomic sentence. Let $A \vee \neg A$ fail to hold at w' (so to speak). The stability axioms hold in this model, and the frame is directed. But $\Diamond A$ holds at w, but A does not (invoking a classical meta-meta-language, so that we can say this).

I think we do have a nice analogue of Øystein's Lemma 5.3 however. As usual, say that a formula Φ is *decidable* in a given theory if

$(\Phi \vee \neg\Phi)$ is provable in that theory.

> LEMMA: Let Φ be a fully modalized formula in a modal language. Then intuitionistic S4.2, the stability axioms for that language, *and the decidability* of all atomic formulas, prove that $\Diamond\Phi$, Φ, and $\Box\Phi$ are equivalent.

Note: In many of the cases of interest, the atomic formulas are decidable. In intuitionistic arithmetic, for example, identity is decidable (as shown by an easy induction). Arguably, the same would hold in intuitionistic analogues of our (fully) Aristotelian theories. Or we can stipulate that it does.

PROOF sketch: Clearly, as with Øystein's proof, it suffices to show that if Φ is fully modalized, then $\Diamond\Phi \rightarrow \Box\Phi$ holds. The proof goes by induction on the complexity of Φ. If Φ is atomic, then Øystein's own argument goes through, since (by hypothesis) Φ is decidable: Suppose $\Diamond\Phi$. We also have $\Phi \vee \neg\Phi$. Argue by cases. If Φ then, by stability, we have $\Box\Phi$, and we are done. If $\neg\Phi$, then, by stability $\Box\neg\Phi$. But this is inconsistent with the hypothesis $\Diamond\Phi$.

Now for the induction steps. We don't really need to do the case for negation, since the intuitionist takes negation to be defined. But we will anyway, just for the fun of it. Suppose the Lemma holds for Φ. Quick and dirty argument: Suppose $\Diamond\neg\Phi$. So $\neg\Box\Phi$. By the contrapositive of the induction hypothesis, $\neg\Diamond\Phi$. So $\Box\neg\Phi$. But, the duality of $\Diamond$ and $\Box$ is problematic (although I am pretty sure about the inferences just invoked). But to be sure: Suppose that $\Diamond\neg\Phi$ holds at a given world w. We want to show that $\Box\neg\Phi$ also holds at w. So suppose $w \leq w'$. Suppose, for reductio, that Φ holds at w'. Then, by the induction hypothesis, $\Box\Phi$ holds at w'. Since $\Diamond\neg\Phi$ holds at w, there is a world w'' such that $w < w''$ and $\neg\Phi$ holds at w''. By the contrapositive of the induction hypothesis, $\neg\Diamond\Phi$ holds at w". By directness, there is a world w''' such that $w' \leq w'''$ and $w'' \leq w'''$. By the former, we have that Φ holds at w''' and so $\Diamond\Phi$ holds at w''. This is a contradiction. So Φ does not hold at w'. So $\neg\Phi$ holds at w'. Since w' is arbitrary, $\Box\neg\Phi$ holds at w. (Note: the quick and dirty argument does not use directness; this one does.)

Now suppose that the Lemma holds for Φ and for Ψ. Suppose $\Diamond(\Phi\&\Psi)$. Then $\Diamond\Phi\&\Diamond\Psi$. So, by the induction hypothesis, $\Box\Phi\&\Box\Psi$. So $\Box(\Phi\&\Psi)$.

Suppose, again, that the Lemma holds for Φ and for Ψ. Suppose $\Diamond(\Phi \vee \Psi)$. Then $\Diamond\Phi \vee \Diamond\Psi$. So, by the induction hypothesis, $\Box\Phi \vee \Box\Psi$. So $\Box(\Phi \vee \Psi)$.

Suppose, for the third time, that the Lemma holds for Φ and for Ψ. Let w be a world in an interpretation, and suppose that $\Diamond(\Phi \rightarrow \Psi)$ holds at w. We want to show that $\Box(\Phi \rightarrow \Psi)$ holds at w. So suppose that $w \leq w'$ and Φ holds at w'. We want to show that Ψ holds at w'. We have that $\Diamond\Phi$ holds at w. So, by the induction hypothesis, $\Box\Phi$ holds at w. We have that $\Diamond(\Phi \rightarrow \Psi)$ holds at w. So there is a world v such that $w \leq v$ and $(\Phi \rightarrow \Psi)$ holds at v. Also Φ holds at v (since $\Box\Phi$ holds at w). So Ψ holds at v. And so $\Diamond\Psi$ holds at the original world w. By the induction hypothesis, $\Box\Psi$ holds at w. So Ψ holds at w' (since $w \leq w'$). So $(\Phi \rightarrow \Psi)$ holds at w' (discharging the assumption that Φ holds at w). But w' is arbitrary. So $\Box(\Phi \rightarrow \Psi)$ holds at w.

Now suppose that the Lemma holds for each instance of $\Phi(x)$. We have to show that it holds for $\Box\forall x\Phi(x)$. So assume that $\Diamond\Box\forall x\Phi(x)$ holds at a world w. We have to show that $\Box\Box\forall x\Phi(x)$ holds at w. This is the same as showing that $\Box\forall x\Phi(x)$ holds at w. So suppose that $w \leq w'$ and let a be an object that exists at w' (so that $a \in D(w')$). By the (G) principle, we have that $\Box\Diamond\forall x\Phi(x)$ holds at w and so $\Diamond\forall x\Phi(x)$ holds at w'.

So there is a world w'' such that $w' \leq w''$ and $\forall x \Phi(x)$ holds at w''. Since the domains grow (or do not shrink) along the accessibility relation, our object a exists at w'', and so $\Phi(a)$ holds at w''. So $\Diamond\Phi(a)$ holds at w'. By the induction hypothesis, $\Phi(a)$ holds at w'. Since a was arbitrary, we have that $\forall x \Phi(x)$ holds at w'. Since w' was arbitrary, we have that $\Box\forall x \Phi(x)$ holds at w. And by S4, $\Box\Box\forall x \Phi(x)$ holds at w.

Suppose, again, that the Lemma holds for each instance of $\Phi(x)$. We have to show that it holds for $\Diamond\exists x \Phi(x)$. So assume that $\Diamond\Diamond\exists x \Phi(x)$ holds at a world w. In S4, this amounts to assuming that $\Diamond\exists x \Phi(x)$ holds at w. We have to show that $\Box\Diamond\exists x \Phi(x)$ holds at w. So suppose that $w \leq w'$. We have to show that $\Diamond\exists x \Phi(x)$ holds at w'. Since $\Diamond\exists x \Phi(x)$ holds at w, there is a world v such that $w \leq v$ and $\exists x \Phi(x)$ holds at v. So there is an object a in $D(v)$ such that $\Phi(a)$ holds at v. By the induction hypothesis, $\Box\Phi(a)$ holds at v. By directedness, there is a world w'' such that $w' \leq w''$ and $v \leq w''$. Since $\Box\Phi(a)$ holds at v, $\Phi(a)$ holds at w''. So $\exists x \Phi(x)$ holds at w''. So $\Diamond\exists x\Phi(x)$ holds at w'. Since w' is arbitrary, we have that $\Box\Diamond\exists x\Phi(x)$ holds at w.

Suppose that the Lemma holds for Φ. We have to show that it holds for $\Box\Phi$. So assume that $\Diamond\Box\Phi$ holds at a world w. By the (G) principle, $\Box\Diamond\Phi$ holds at w. Let $w \leq w'$. Then $\Diamond\Phi$ holds at w'. By the induction hypothesis, $\Box\Phi$ holds at w'. Since w' is arbitrary, $\Box\Box\Phi$ holds at w.

Suppose again (and finally) that the Lemma holds for Φ. We have to show that it holds for $\Diamond\Phi$. So assume that $\Diamond\Diamond\Phi$ holds at a world w (to show that $\Box\Diamond\Phi$ holds at w). We thus have that $\Diamond\Phi$ holds at w. By the induction hypothesis, $\Box\Phi$ holds at w. Let $w \leq w'$. So Φ holds at w'. Since the accessibility relation is reflexive, we have that $\Diamond\Phi$ holds at w'. Since w' is arbitrary, we have that $\Box\Diamond\Phi$ holds at w.

This completes the proof of the Lemma.

Here is one statement of our intuitionistic Mirroring Theorem:

> Let $\vdash$ be the relation of intuitionistic deducibility in a given language (although, as in Øystein's paper, if the language is plural, we don't include any comprehension axioms). Let $\vdash^{\Diamond}$be the corresponding deductibility relation in intuitionistic S4.2, the stability axioms, *and the decidability* of all atomic formulas. Let $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_n$, and $\mathbf{\Psi}$ be non-modal formulas. Then

$$\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_n, \vdash \mathbf{\Psi} \text{ if and only if } \mathbf{\Phi}_1^{\Diamond}, \ldots, \mathbf{\Phi}_n^{\Diamond} \vdash^{\Diamond} \mathbf{\Psi}^{\Diamond}.$$

Note: We are being a little sloppy here (and, in a few places, in the proof of the Lemma). The intuitionistic deductive system in the Simpson dissertation requires symbols for the worlds. So the right hand side should read that for all worlds w, if the premises hold in w, then so does the conclusion. We can suppress mention of the world for almost all of the cases (I think).

Proof sketch. Start with the left-to right direction. As in Øystein's classical version, the proof goes by induction on the derivation. As Øystein notes, the only hard cases are the quantifier rules, and the proof he gives for the universal quantifier goes straight through (technically by adding mention of a world). But just to be sure:

For the universal elimination rule, suppose we have $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_n, \vdash \forall x \mathbf{\Psi}(x)$. The induction hypothesis gives us $\mathbf{\Phi}_1^{\Diamond}, \ldots, \mathbf{\Phi}_n^{\Diamond} \vdash^{\Diamond} \Box \forall x \mathbf{\Psi}^{\Diamond}(x)$, and from this conclusion we get $\forall x \mathbf{\Psi}^{\Diamond}(x)$ and thus $\mathbf{\Psi}^{\Diamond}(t)$.

For the universal introduction rule, suppose we have $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_n, \vdash \mathbf{\Psi}(t)$, where the t does not occur free in any of the premises. By the induction hypothesis, we have $\mathbf{\Phi}_1^{\Diamond}, \ldots, \mathbf{\Phi}_n^{\Diamond} \vdash^{\Diamond} \mathbf{\Psi}^{\Diamond}(t)$. And so $\mathbf{\Phi}_1^{\Diamond}, \ldots, \mathbf{\Phi}_n^{\Diamond} \vdash^{\Diamond} \forall x \mathbf{\Psi}^{\Diamond}(x)$. A standard move in S4 gives us $\Box \mathbf{\Phi}_1^{\Diamond}, \ldots, \Box \mathbf{\Phi}_n^{\Diamond} \vdash^{\Diamond} \Box \forall x \mathbf{\Psi}^{\Diamond}(x)$. Our Lemma thus gives us $\mathbf{\Phi}_1^{\Diamond}, \ldots, \mathbf{\Phi}_n^{\Diamond} \vdash^{\Diamond} \Box \forall x \mathbf{\Psi}^{\Diamond}(x)$. The conclusion of this is $(\forall x \mathbf{\Psi}(x))^{\Diamond}$.

For the existential introduction rule, suppose we have $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_n, \vdash \mathbf{\Psi}(t)$. The induction hypothesis gives us

$\Phi_1^\Diamond, \ldots, \Phi_n^\Diamond \vdash^\Diamond \Psi^\Diamond(t)$, and so we have $\Phi_1^\Diamond, \ldots, \Phi_n^\Diamond \vdash^\Diamond \exists x \Psi^\Diamond(x)$, and so $\Phi_1^\Diamond, \ldots, \Phi_n^\Diamond \vdash^\Diamond \Diamond\exists x \Psi^\Diamond(x)$.

For the existential elimination rule, we do have to mention the worlds in question (I think). Suppose we have $\Phi_1, \ldots, \Phi_n, \Phi(t) \vdash \Psi$, where t does not occur free in any of the Φ_i, nor in Ψ. The induction hypothesis thus gives us $\Phi_1^\Diamond, \ldots, \Phi_n^\Diamond, \Phi^\Diamond(t) \vdash^\Diamond \Psi^\Diamond$. That is, any world in which the premises hold, the conclusion holds as well (in that world). We have to show that $\Phi_1^\Diamond, \ldots, \Phi_n^\Diamond, \Diamond\exists x \Phi^\Diamond(x) \vdash^\Diamond \Psi^\Diamond$. So let w be any world in which $\Phi_1^\Diamond, \ldots, \Phi_n^\Diamond$,and $\Diamond\exists x \Phi^\Diamond(x)$ all hold. We have to show that $\Psi^\Diamond$ holds at w. Since $\Diamond\exists x \Phi^\Diamond(x)$ holds at w, there is a world w' such that $w \leq w'$ and $\exists x \Phi^\Diamond(x)$ holds at w'. So there is an object a in the domain of w' such that $\Phi^\Diamond(a)$ holds at w'. By the above Lemma, we have that $\Box\Phi_1^\Diamond, \ldots,$ and $\Box\Phi_n^\Diamond$ all hold at w. So $\Phi_1^\Diamond, \ldots,$ and $\Phi_n^\Diamond$ all hold at w'. Now recall that the induction hypothesis gives us $\Phi_1^\Diamond, \ldots, \Phi_n^\Diamond, \Phi^\Diamond(t) \vdash^\Diamond \Psi^\Diamond$. The premises all hold at w', interpreting t as a. So we have that $\Psi^\Diamond$ holds at w'. So, since $w \leq w'$, we have that $\Diamond\mathbf{\Psi}^{\Diamond}$ holds at the original world w. By the Lemma, we have that $\mathbf{\Psi}^{\Diamond}$ holds at w. This is what we were to show.

Now for the right-to-left direction. Øystein gives us an easy proof of the converse of the above mirroring theorem, in the classical context:

> The right-to-left direction is easily established by adopting a Hilbert-style axiomatic approach to the relevant modal logic. Consider the operation $\Phi \to \Phi^{-}$ of deleting all modal operators. This operation maps every axiom of our modal logic to a theorem of the corresponding non-modal logic and correlates every inference rule of the former with a legitimate inference of the latter. The right-to-left direction [of the mirroring theorem] then follows from the observation that $(\Phi^{\Diamond})^{-} = \Phi$.

Something similar holds here. Take any derivation of $\mathbf{\Phi}_1^\Diamond, \ldots, \mathbf{\Phi}_n^\Diamond \vdash^\Diamond \mathbf{\Psi}^\Diamond$, and erase all of the boxes and diamonds, and all of the parameters for worlds. The result is easily converted in to an intuitionistic derivation of $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_n, \vdash \mathbf{\Psi}$.

4.3 Why the Mirroring Theorem is available

Provided that the assumptions of the Mirroring Theorem are satisfied, this bridge makes it straightforward to 'potentialize' the Aristotelian theories A^3 and A^{3^*} developed above. The Theorem will ensure that any reasoning from our actualist Aristotelian theories can be duplicated in the potentialist setting. All that remains is therefore to verify that the Mirroring Theorem is in fact available.

We have already argued above (§4.1) that the modal logic is (at least) S4.2. The first task is to defend the stability of our primitive predicates: $=$, $\leq$, L, $Cong$, and, in A^{3^*}, $\prec$.

We begin with the geometric primitives. Recall that $L(x,y)$ means that region x is completely to the left of region y. The stability of this predicate comes to

$$L(x,y) \rightarrow \Box L(x,y)$$
$$\neg L(x,y) \rightarrow \Box \neg L(x,y)$$

These principles would obviously be false if the variables were given unrestricted range. Consider, for instance, two sticks on a line. Although one stick is (completely) left of the other, it might have been the other way round (and we can make it the case that it is the other way around by moving one of the sticks).

But recall that our (first-order) variables are meant to range over *regions*. And it is plausible to take regions to be individuated, at least in part, by their geometrical properties. Thus, when one region is (or is not) entirely left of another, there is no construction we can undertake that would change this fact. In effect, regions do not move (at least not relative to other regions). This motivates the pair of stability principles displayed above. The case for the stability of $Cong$ is entirely analogous. If two intervals are congruent (or not), then no construction can change that. Constructions do not change the length of intervals.

The stability of identity was famously defended by Kripke [1971], and we can simply adopt his argument. In light of the foregoing, however, the case here is especially strong, and does not depend so much on intuitions as to what is metaphysically possible concerning things like people, sticks, and houses. If two *regions* are identical (or distinct), then, intuitively, they remain identical (or distinct) even as various constructions are performed. The same goes for the stability of $\leq$, the mereological parthood relation.

We adopt the policy (or view) of Linnebo [2013] **(**cf. also Uzquiano [2012]**)** that pluralities are determined entirely by their "members". So if an object a is (resp. is not) one of the bb in one world, then a is (resp. is not) one of the bb in every world that contains both a and the bb. This policy is especially clear in the present, Aristotelian context, since we only envision finite pluralities. We can just stipulate that such pluralities are "individuated" by the objects they "contain".

That takes care of the stability of the primitives. Note that in the development of the "potentialist" theory, the various definitions have to be "modalized". For example, in the above semi-Aristotelian and Aristotelian theories, we defined two regions x, y to overlap, written $x \circ y$, just in case there is a region that is part of both: $\exists z(z \leq x \,\&\, z \leq y)$. Here, we do not want to say that. Intuitively, two overlapping regions might exist in a given world, even if there is no region *in that world* that is part of both. What we want to say is that two regions overlap just in case there *could be* a region that is part of both. So $x \circ y$ is defined to be $\Diamond\exists z(z \leq x \,\&\, z \leq y)$. Similarly, to say that x and y are discrete, written $x|y$, is to say that it is not possible for there to be a region that is part of both: $\neg\Diamond\exists z(z \leq x \,\&\, z \leq y)$.

It is, of course, a routine syntactic matter to "modalize" the other definitions. We give ordinary language glosses on some of them, to help motivate the present orientation. Recall that a region x is *connected* if $\forall y, z, u[z, u \leq x$ & $Betw(z, y, u) \rightarrow y \leq x]$. Modalized, this says that necessarily, for any regions y, z, u, if y and z are parts of x and if u is Between y and z, then u, too, is part of x. A region is *bounded* if there could be a region to its left and there could be a region to its right. Two regions are *adjacent* if there cannot be a region that is part of both, and there cannot be a region that is between them. Finally, recall that two intervals j, k are *left-end equivalent* if $\exists p[p \leq j$ & $p \leq k$& $\nexists q(\{q \leq j \vee q \leq k\}$ & $L(q, p)]$. Modalized, this says that there could be region p that is part of both, such that there cannot be a region that is part of either j or k that is also Left of p. We take that these are all what we want.

Our final task is to verify that the potentialist translations of the axioms of A^3 and A^{3^*} are acceptable to the potentialist. The potentialist translation of axiom 1a says that necessarily, $\leq$ is a partial order. For instance, the transitivity axiom is necessitated:

$$\Box\forall x\forall y\forall z(x \leq y \,\&\, y \leq z \to x \leq z) \tag{1}$$

This is acceptable to the potentialist for much the same reason as the stability axioms considered above; and the same goes for the reflexivity and anti-symmetry claims.

The next axiom is 1b, which says that $x \leq y$ iff anything that overlaps x also overlaps y. If we unpack the definitions, as above, and then modalize, the result is the necessitation of the universal closure of the following:

$$x \leq y \leftrightarrow \Box\forall z(\Diamond\exists w(w \leq z \,\&\, w \leq x) \to \Diamond\exists w(w \leq z \,\&\, w \leq y)$$

This is indeed acceptable to the potentialist: x is part of y if and only if any region z that we may generate possibly has a joint part with x only if that same z possibly has a joint part also with y.

Recall that, in effect, in our semi-Aristotelian theory A^3, the unrestricted fusion axiom has been replaced by (FIN-FUS), which states that any two regions have a fusion:

$$\forall u \forall v \exists x \forall y [y \circ x \leftrightarrow (y \circ u) \vee (y \circ v)].$$

Modalized, this comes to:

$$\Box \forall u \forall v \Diamond \exists x \Box \forall y [y \circ x \leftrightarrow (y \circ u) \vee (y \circ v)].$$

In words, this says that, necessarily, for any regions u, v, there could be a region x (i.e., $u + v$), such that, necessarily, any region can share a part with v just in case that either that region can share a part with u or that region can share a part with v. This, we submit, is what what we want.

The alternate system A^{3^*} has, instead, an axiom, which states that any plurality—which according to our Aristotelian will be finite—has a fusion. The potentialist translation of this is

$$\Box\forall xx\Diamond\exists y\,(y = \sum xx)^{\Diamond} \qquad \text{(P-FUS}^{\Diamond}\text{)}$$

where $(y = \sum xx)^{\Diamond}$ in turn unpacks as:

$$\Box\forall z(\Diamond\exists w(w \prec z \,\&\, q \prec y) \leftrightarrow \Diamond\exists x(x \prec xx \,\&\, \Diamond\exists w(w \prec z \,\&\, w$$

Although this latter formula is quite a mouthful, we contend that, from the potentialist's point of view, this is a perfectly plausible definition of the claim that y is the fusion of xx. Accordingly, (P-FUS$^{\Diamond}$) says what the potentialist wishes to say, namely that necessarily, any regions possibly have a fusion.

Recall our axioms (FIN-P1) and (FIN-P2) that govern finite pluralities. The first says that for each object x there is a plurality that contains just x and the second states that the result of adding a object to plurality results in another plurality. The modalizations of these are, of course, acceptable. We could just assert the necessitations of these. For example, we don't have to think of the existence of a plurality that contains just x requires a separate construction. So we could say, instead, that necessarily, for each region x there is a plurality that contains just x.

Consider, next, our (Euclidean) axiom that attempts to state the finitude of each world. Modalized, this axiom says that, necessarily, if the xx are among the yy, but the xx are not all of the yy, then the xx are not equinumerous with the yy. Again, this seems correct.

Since, on the intended interpretation, all worlds are finite, we can replace (FIN-P1) and (FIN-P2) with the (necessitation of the) ordinary (non-modal) plural comprehension scheme:

$$\exists y\phi(y) \rightarrow \exists xx\forall y(y \prec xx \leftrightarrow \phi(y)) \qquad \text{(P-Comp)}$$

for any ϕ without xx free. The reason is that if every "world" is indeed finite, then there is no difference between the pluralities defined by any non-empty condition $\phi(y)$ and those defined by finitely many steps of adding one object.

It is important not to confuse the non-modal plural comprehension scheme displayed above with *the potentialist translation* of the same scheme:

$$\Diamond\exists y\phi(y) \to \Diamond\exists xx\Box\forall y(y \prec xx \leftrightarrow \phi^{\Diamond}(y)) \qquad (\text{P-Comp}^{\Diamond})$$

This we do not want. For instance, the instance of this scheme where ϕ is $y = y$ is false for our Aristotelian: for there will *never be* a plurality—and thus a completed collection—of all self-identical objects. The principle P-Comp$^{\Diamond}$ would yield a modalized version of the unrestricted fusion principle, Axiom 2, of the semi-Aristotelian theory. Of course we do not want this—it makes our potential infinities actual, and our Aristotelian assumes that *no* potential infinities can be completed. P-Comp$^{\Diamond}$ cannot be obtained from P-Comp, as can be seen with a Kripke structure in which every "world" is finite.

We conclude this section by giving glosses on the modalized versions of the remaining axioms. First, the axioms to the semi-Aristotelian theory that carry over:

> Axiom 3: Necessarily, if a region x is either Left or Right of a region y then x and y are discrete (there cannot be a part of both). And, necessarily, x is Left of y just in case, necessarily, every part of x is Left of every part of y.
>
> Axiom 4: (Dichotomy) Necessarily, if i and j are discrete intervals, then either i is Left of j or i is Right of j.
>
> Axiom 5 (Gunkiness): Necessarily, any region can have an interval as a proper part.
>
> Axiom 6 (Translation): Necessarily, for any intervals i, j, there could be an interval k that is congruent with i and right-end equivalent with j; and there could be an interval k' that is congruent with i and left-end equivalent to j.

> Axiom 7 (Additivity): Necessarily, for any intervals i, j, i', j', if i is Adjacent to j and i' is Adjacent to j', and if i is congruent to i' and j is congruent to j', then, necessarily, the fusion of i and j is congruent to the fusion of i' and j'.

This last is, again, a mouthful, but we take it that it is clearly acceptable.

Finally, here are the glosses on the remaining axioms of A^3 and A^{3^*}:

Differences: Necessarily, for any regions x, y, if there could be a region that overlaps x and is discrete from y, then there can be a region w such that, necessarily, any region is a part of w just in case it is part of x and discrete from y.

(*n*-SECT): Necessarily, for any interval i there can be intervals $j_1, \ldots, j_n$ which are congruent to each other and pairwise discrete, such that i is the fusion of them.

(*n*-SECT*): Necessarily, for any interval i and any objects pp, there could be some intervals jj such that the jj are equinumerous with the pp, the jj are pairwise discrete, the jj are all congruent to each other, and i is the fusion of the jj.

(ARCH2): Necessarily, for any interval i and any region k, there is (or could be) a natural number n such that k is part of an n-fold biextension of i.

(ARCH 2*): Necessarily, for any interval i and any region k, there could be some things xx such that k is part of an xx-fold biextension of i.