The role of diagrams in mathematics

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Content

- On the role of visualisation in mathematics.

  In particular role of diagrams.

- Concrete examples displaying various roles.

- The role of diagrams: Forming concepts and proofs.

- The role of diagrams: Display - or representation - of relations.

Diagrams in mathematics

Throughout history diagrams have played a significant role in mathematics. E.g.,

- Euclid's Elements:

  Netz (1999) argues that diagrams are metonyms for propositions.

  More recently Catton and Montelle (2012) argue that "the Greek mathematical diagram is metonym for an epiphany".

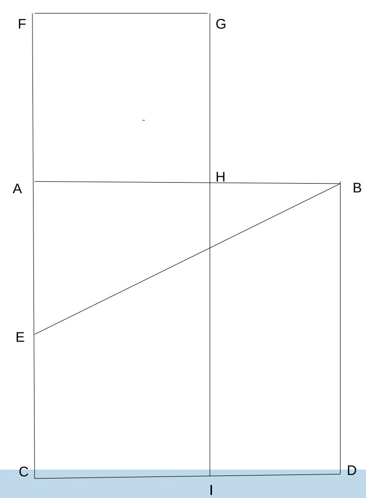
- Analysis founded on geometry.

Diagrams in mathematics

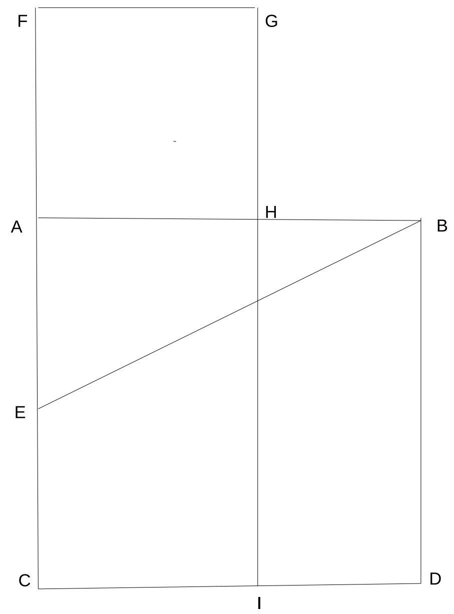
Already in Euclid, we get some answers as to which role diagrams play.

Take as an example Euclid's proposition II.11:

To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment.

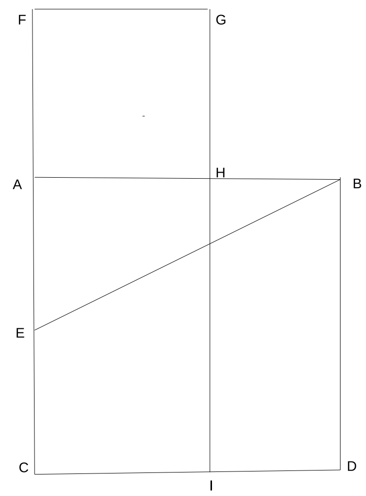


Diagrams in mathematics



AB is the given line.
$\square AB$ is constructed, E is marked, so that $AE = EC$
Draw EB, and make $EF = EB$.
Construct $\square AF, FG$. H is the sought for point.

Diagrams in mathematics



Part of proof.

It is shown that rectangle FI equals $\square AD$.

The diagram then shows that

$\square AH =$rectangle HB, AB.

Diagrams in mathematics

We may thus already point to three roles that diagrams play:

1. The diagram helps *recalling* the (formulation of the) proposition (Netz).

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Diagrams in mathematics

We may thus already point to three roles that diagrams play:

1. The diagram helps *recalling* the (formulation of the) proposition (Netz).
2. The diagram indicates the construction and furthermore contributes to the *understanding* of the demonstration (Catton and Montelle).
3. The diagram sometimes even contributes to the actual proof of the proposition (Manders, Mumma).

Diagrams in mathematics

During 19th century diagrams - or pictures - became discredited.

Analysis moved towards an arithmetic foundation. Eg., concepts of *function*, *continuity* and *differentiability* were developed so that it is possible to form the geometrically unintuitive "continuous, but nowhere differentiable, function".

Even in geometry diagrams were disapproved of, here in the famous words by Pasch:

" For the appeal to a figure is, in general, not at all necessary. It does facilitate essentially the grasp of the relations stated in the theorem and the constructions applied in the proof. Moreover, it is a fruitful tool to discover such relationships and constructions. However, if one is not afraid of the sacrifice of time and effort involved, then one can omit the figure in the proof of any theorem; indeed, the theorem is only truly demonstrated if the proof is completely independent of the figure". (Pasch, 1882/1926, 43).
(Quote from Mancosu 2005.)

(Philosophical) discussion on the role of diagrams - and visualisation

Much focus has been on the question whether diagrams can be used to obtain rigorous proofs.

- Diagrams do not provide rigorous proofs, since
  - In Euclid and in analysis: We **have been** deceived in the past!
  - General properties read off from particular instances.
- Manders defends/explains the use of diagrams in Euclid (exact and non-exact properties)
- Diagrams *can* be used to give rigorous proofs. (Barwise and Etchemendy 1996, later Azzouni 2013).
- Diagrammatic reasoning is a rigorous and integrated practice in low dimensional geometry (Giardino and Toffoli, forthcoming).

The role of visualisation

Recently focus is also on the many other roles visualisation plays in mathematics — independently of discussion of rigour and proofs.

E.g.

- Past views on visualisation and diagrams: Smadja on Hilbert (2012) and Heis on Projective geometry.
- Role visualisation play for heuristics and understanding (Giaquinto 2005).

The role of visualisation

1. Since visualisation is valued - and play a role in discovery, one may ask **how** visualisation contributes.

2. And: **Why** is visualisation so fruitful?

Tappenden:"*The fact that Riemann surfaces (the basic context for analysis in the Riemann style) allow complex functions to be easily visualized was, and remains, a contributor to the fruitfulness of the Riemann approach.*" (2005, 150).

Roles that diagrams play in proofs.

Case study in part of **Free probability theory**.

Work on the value of a certain expression leads to two types of diagrams:

1. Representation of permutations and
2. Representation of equivalence classes.

The number of equivalence classes depends on whether the permutation is *crossing* or *non-crossing*.

Role of diagrams

I illustrate

1. Diagrams inspire definitions of concepts and proof strategies.
2. They work as mental frameworks or images in part of proofs.

There are two types of visualisation

- as representations, and
- as mental images.

A combinatorial expression in free probability theory

Consider the expression:

$$\mathbb{E} \circ Tr_n[B_1^* B_{\pi(1)} \cdot \ldots B_p^* B_{\pi(p)}] = m^{k(\hat{\pi})} \cdot n^{l(\hat{\pi})},$$

where

- B_i's are Gaussian Random matrices, i.e. matrices with entries complex valued Gaussian random variables.
- $\mathbb{E}$ denotes expectation,
- Tr_n denotes the trace of a matrix.

(Example from U. Haagerup and S. Thorbjørnsen: **Random Matrices and K-theory for exact C^*-algebras**, 1999.)

A permutation $\hat{\pi}$ constructed from π:

We write:

$$B_1^* B_{\pi(1)} B_2^* B_{\pi(2)} \dots B_p^* B_{\pi(p)}$$

in the form

$$C_1^* \cdot C_2 \cdot C_3^* \cdot \ldots \cdot C_{2p-1}^* \cdot C_{2p}.$$

This means that

$$C_{2i-1} = B_i \text{ and } C_{2i} = B_{\pi(i)}.$$

A permutation $\hat{\pi}$ constructed from π:

$$C_{2i-1} = B_i \text{ and } C_{2i} = B_{\pi(i)}.$$

A calculation gives

$$C_{2i-1} = B_i = B_{\pi(\pi^{-1}(i))} = C_{2\pi^{-1}(i)}$$

and

$$C_{2i} = B_{\pi(i)} = C_{2\pi(i)-1}.$$

This gives a permutation $\hat{\pi}$:

$$\begin{aligned} \hat{\pi}(2i-1) &= 2\pi^{-1}(i) \qquad (i \in \{1, 2, \ldots, p\}) \\ \hat{\pi}(2i) &= 2\pi(i) - 1 \quad (i \in \{1, 2, \ldots, p\}) \end{aligned} \tag{1}$$

A permutation $\hat{\pi}$ constructed from π:

One example:

$$B_1^* B_{\pi(1)} B_2^* B_{\pi(2)} B_3^* B_{\pi(3)} B_4^* B_{\pi(4)}$$

is written as

$$C_1^* \cdot C_2 \cdot C_3^* \cdot C_4 \cdot C_5^* \cdot C_6 \cdot C_7^* \cdot C_8.$$

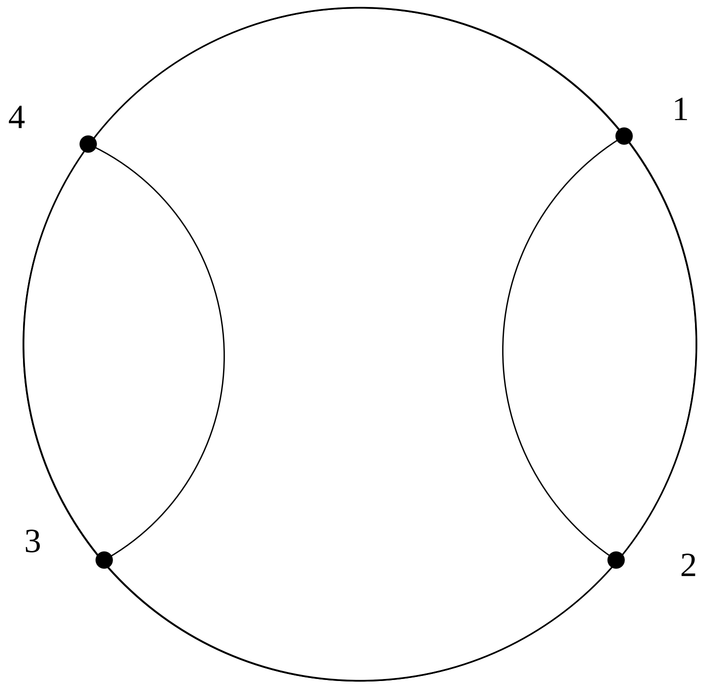
Suppose that $\pi(1) = 2$ and $\pi(3) = 4$, which gives:

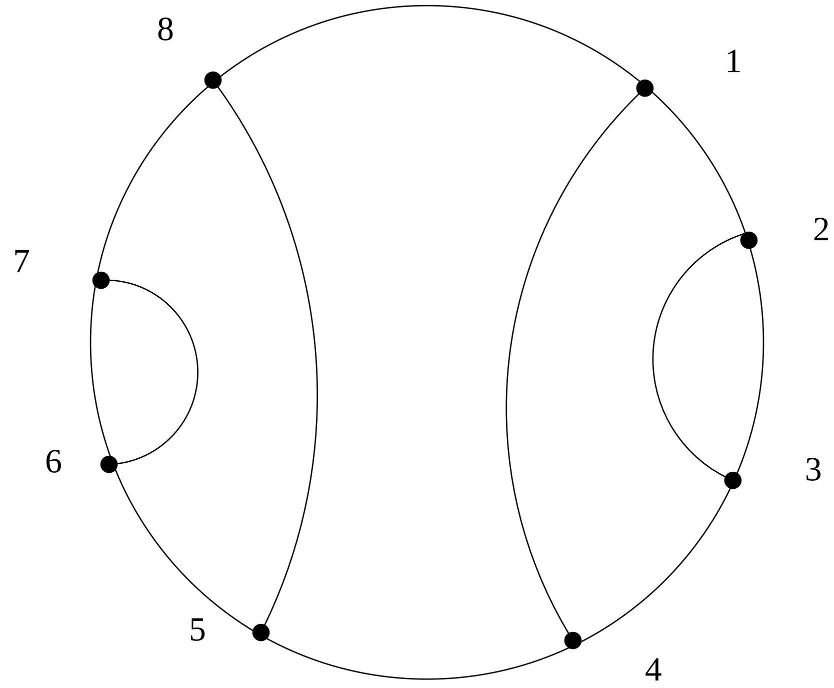
$$B_1^* \cdot B_2 \cdot B_2^* \cdot B_1 \cdot B_3^* \cdot B_4 \cdot B_4^* \cdot B_3$$

This implies that $C_1 = C_4$, $C_2 = C_3$ and so on.

A permutation $\hat{\pi}$ contructed from π:

$C_1 = C_4$, $C_2 = C_3$ ....





Picture 1

π to the left is the permutation $(12)(34)$. $\hat{\pi}$ is shown on the right.

A permutation $\hat{\pi}$ constructed from π:

Another example, given the permutation $(13)(24)$:

$$B_1^* \cdot B_3 \cdot B_2^* \cdot B_4 \cdot B_3^* \cdot B_1 \cdot B_4^* \cdot B_2$$

$$C_1^* \cdot C_2 \cdot C_3^* \cdot C_4 \cdot C_5^* \cdot C_6 \cdot C_7^* \cdot C_8.$$

A permutation $\hat{\pi}$ constructed from π:

$$B_1^* \cdot B_3 \cdot B_2^* \cdot B_4 \cdot B_3^* \cdot B_1 \cdot B_4^* \cdot B_2$$

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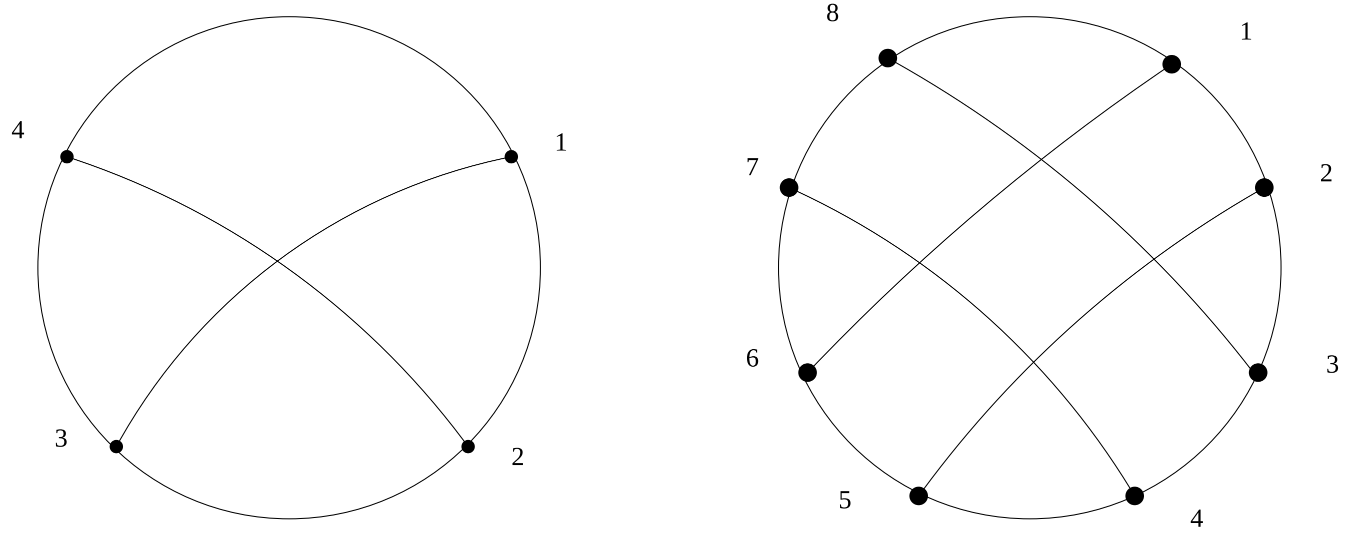


Picture 2

π to the left is the permutation $(13)(24)$. $\hat{\pi}$ is on the right.

Introduction of a relation and equivalence classes.

In the next step the product of the matrices is calculated:

$$\sum_{\substack{1\leq u_1,u_3,\ldots u_{2p-1}\leq n\\ 1\leq u_2,u_4,\ldots u_{2p}\leq m}} \mathbb{E}[\overline{b(u_2,u_1,1)}b(u_2,u_3,\pi(1))\cdots\overline{b(u_{2p},u_{2p-1},p)}b(u_{2p},u_1,\pi(p))].$$

From this expression it is possible to deduce a condition for the expression to be $\neq 0$:

$$b(u_{2i},u_{2i+1},\pi(i)) = b(u_{2\pi(i)},u_{2\pi(i)-1},\pi(i)), \quad (i\in\{1,2,\ldots,p\})$$

or

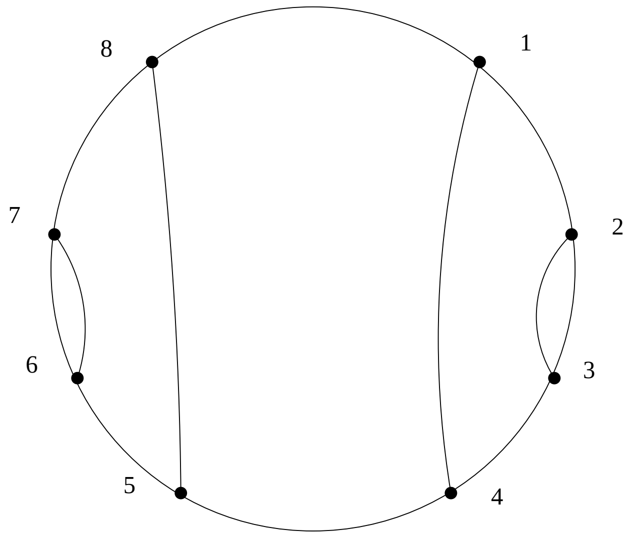
$$u_{2i} = u_{2\pi(i)} \quad \text{and} \quad u_{2i+1} = u_{2\pi(i)-1}$$

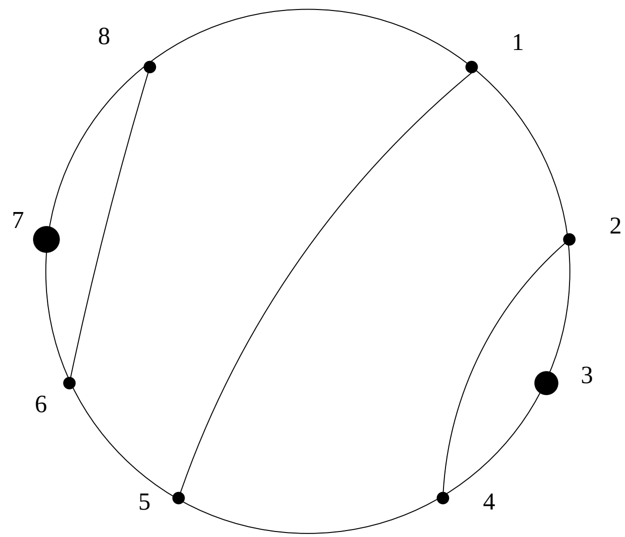
($b(i,j,k)$ denotes the (i,j)'th entry of B_k)

Introduction of a relation and equivalence classes

These equations lead to an equivalence relation on the set $\{1, 2, \ldots, 2p\}$:

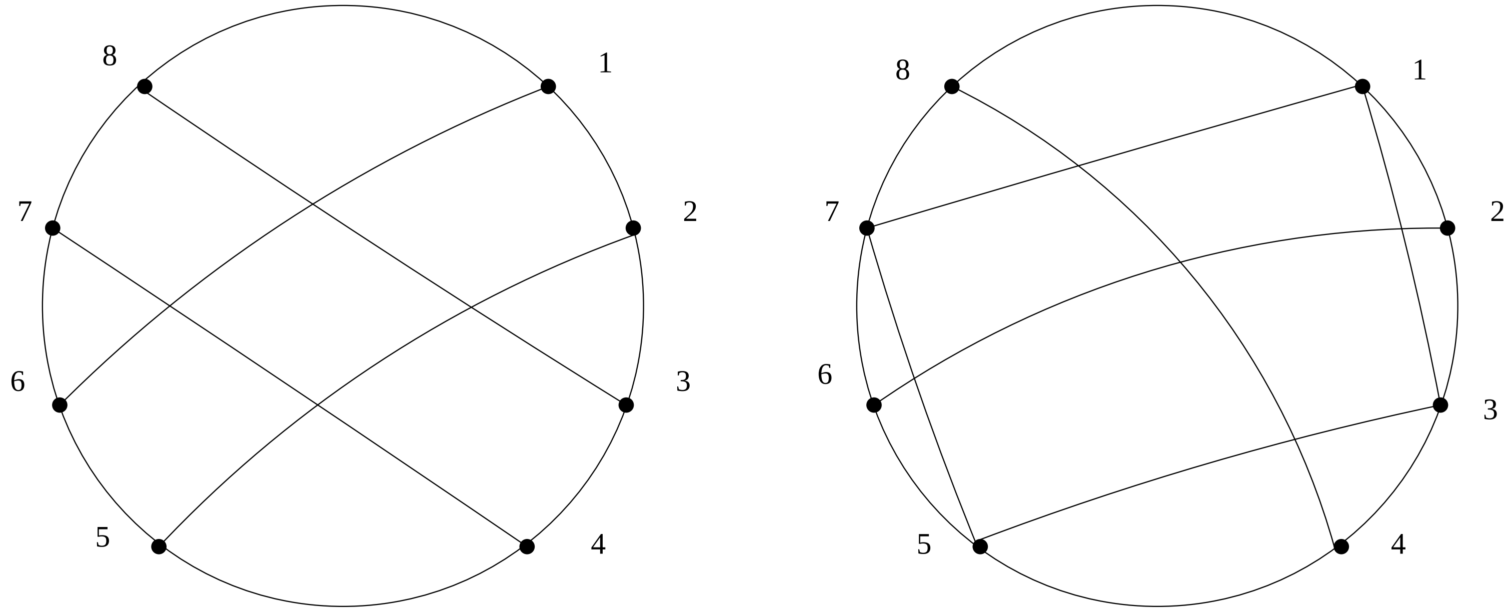
$$j \sim_{\hat{\pi}} \hat{\pi}(j) + 1$$





$\hat{\pi}$ from picture 1 is pictured to the left. Equivalence classes contain numbers that are joined by lines in the diagram on the right.

Introduction of a relation and equivalence classes



$\hat{\pi}$ from picture 2 is on the left. Equivalence classes contains numbers that are joined by lines in the diagram on the right.

Note that the the equivalence classes contain either only **even** numbers or **odd** numbers.

The value of the expression

$$\mathbb{E} \circ Tr_n[B_1^* B_{\pi(1)} B_2^* B_{\pi(2)} \dots B_p^* B_{\pi(p)}] = m^{k(\hat{\pi})} \cdot n^{l(\hat{\pi})}$$

$k(\hat{\pi})$ denotes the number of equivalence classes with even numbers and $l(\hat{\pi})$ denotes the number of equivalence classes with odd numbers.

The role of diagrams:

I show examples illustrating two roles that diagrams play:

1. Diagrams inspire new concepts and proof strategies.

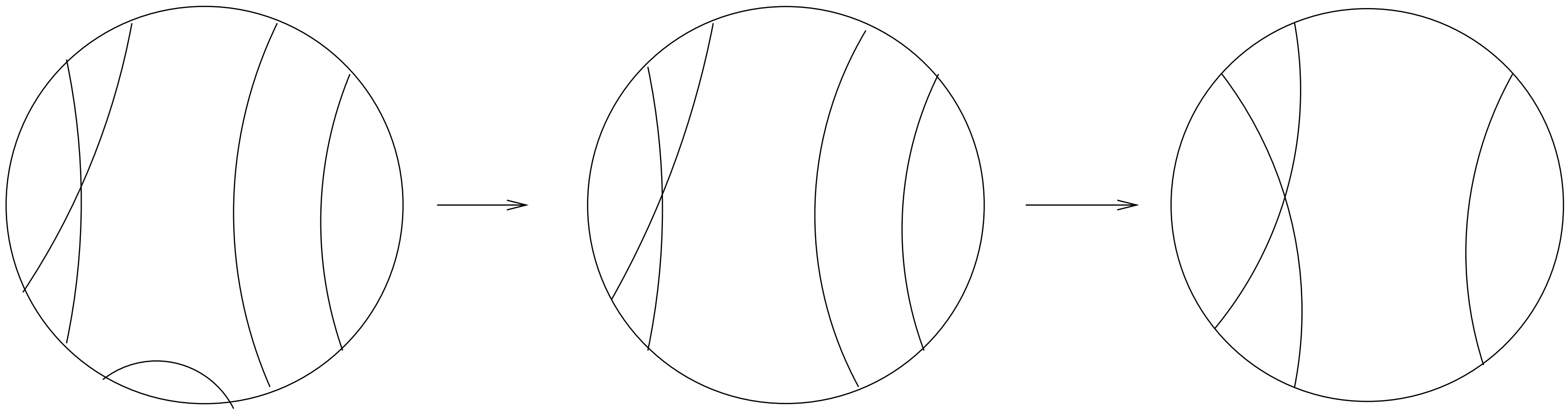
2. They work as mental pictures in part of proofs.

The examples come from a proof of the following theorem:

Let p be a positive integer and $\pi \in S_p$. Then $\hat{\pi}$ is non-crossing if and only if $k(\hat{\pi}) + l(\hat{\pi}) = p + 1$.

Role of diagrams:

A sketch of the proof:



1. With each removal of a pair, you remove an equivalence class.

2. The trivial permutation has one equivalence class.

The role of diagrams:

Supporting 1, a number of definitions are inspired from the diagrams:

- non-crossing and crossing permutations
- (cancelling of) neighbors.

Furthermore there are proofs that easily can be visualised. For example:

The role of diagrams:

Let p be a positive integer and $\pi \in S_p^{nc}$. Then $\hat{\pi}$ has a pair of neighbors.

Definition. A pair of neighbors: Let π be a permutation in S_p, and let e be an element of $\{1, 2, \ldots, 2p-1\}$. We say that $(e, e+1)$ is a pair of neighbors for $\hat{\pi}$ if $\hat{\pi}(e) = e + 1$.

Definition. A crossing permutation: Let π be a permutation in S_p. We say that $\hat{\pi}$ is a crossing permutation, if for some $a < b < c < d$ in $\{1, 2, \ldots, 2p\}$ is the case that $\hat{\pi}(a) = c$ and $\hat{\pi}(b) = d$.

The proof then is:
Proof. It is proved that if $\pi \in S_p$ has no pair of neighbors, then π has a crossing.
Consider the set $M = \{j \in \{1, 2, \ldots, 2p\} : \hat{\pi}(j) \geq j\}$. Note that $M \neq 0$, since clearly $1 \in M$. Define now

$$\alpha = \min\{\hat{\pi}(j) - j : j \in M\}.$$

The role of diagrams:

Since $\hat{\pi}$ has no fixed points and no pairs of neighbors, we must have $\alpha \geq 2$. Choose $j \in \{1, 2, \ldots, 2p\}$ so that $\hat{\pi}(j) - j = \alpha$.

Since $\alpha \geq 2$, $\hat{\pi}(j) \neq j + 1$, or equivalently $\hat{\pi}(j+1) \neq j$. Combining this with the definition of α, and the fact that $\hat{\pi}$ has no fixed points, it follows that

$$\hat{\pi}(j+1) \notin \{j, j+1, \ldots j+\alpha\} = \{j, j+1, \ldots, \hat{\pi}(j)\},$$

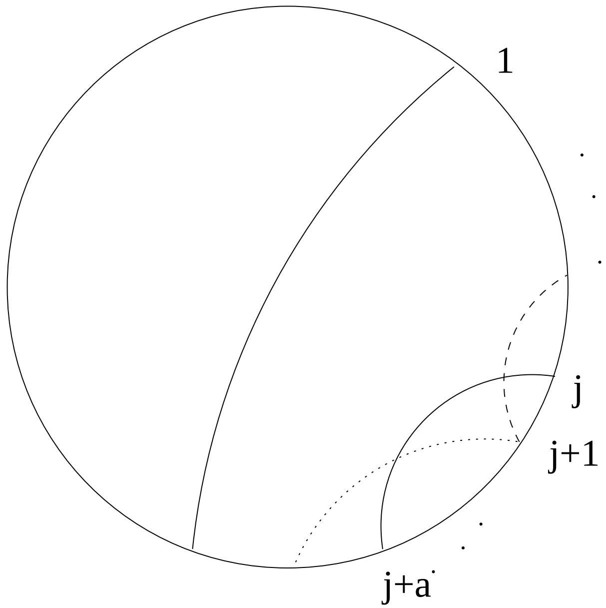
i.e., either $\hat{\pi}(j+1) < j$ or $\hat{\pi}(j+1) > \hat{\pi}(j)$.

In the first case $(\hat{\pi}(j+1), j, j+1, \hat{\pi}(j))$ is a crossing for $\hat{\pi}$. In the second case $(j, j+1, \hat{\pi}(j), \hat{\pi}(j+1))$ is a crossing for $\hat{\pi}$.

In either case, we see that a crossing occurs.

The role of diagrams:

A picture proof:



It is shown that if $\hat{\pi}$ has no pairs, then it has a crossing. j is chosen so that $\hat{\pi}(j) - j$ is as small as possible. Then we consider $\hat{\pi}(j+1)$ and one sees that a crossing must occur.

The role of diagrams:

Claim 2: some proofs depend on an (imaginary) visualisation of the situation:

For example, the proof af the theorem:

Let p be a positive integer and $\pi \in S_p^{nc}$. By finitely many successive cancellations of pairs $\hat{\pi}$ can be reduced to either

i) $\hat{e_1}$, where e_1 is trivial permutation in S_1

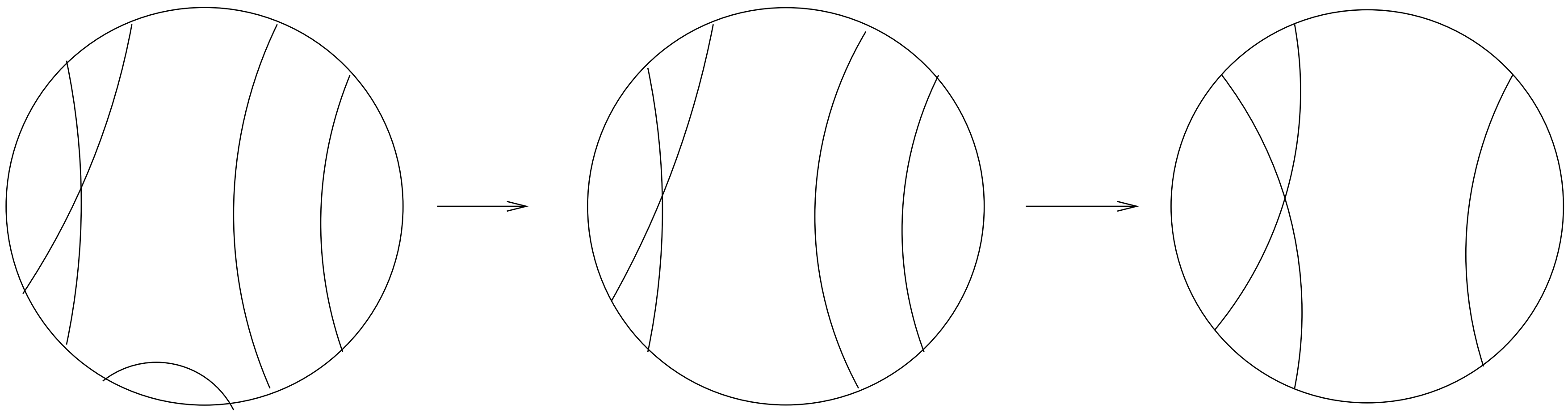
or

ii) $\hat{\rho}$, where ρ is a permutation in S_q^{irr} for some $q \in \{2, 3, \ldots p\}$.

Case i appears iff $\pi \in S_p^{nc}$.

The role of diagrams:

The proof starts with the following claim “It is clear that by successive cancellations we arrive at $\hat{\rho}$ where either $\rho \in S_1$ or ρ is a permutation in S_q^{irr} for some $q \in \{2, 3, \ldots p\}$”.



Successive removals of pairs of neighbors.

Diagrams as representations - display of relations.

Some diagrams we have seen so far *display relations*!

In the semiotics of Peirce the relation between a sign and the object that it represents can be of three kinds, iconic, indexical or symbolic.

The certain kind of icon that represent relations is denoted a diagram.

(According to Peirce all mathematical reasoning is diagrammatic.)

Conclusions - part I

In addition to the roles found in Euclid, we may state the following about the role of diagrams here:

- **Diagrams as representations:** Some uses of visualisation is as diagrams in the sense of Peirce: The crossings and neighbors of the diagrams can be given formal definitions. Using these definitions it is possible to translate the proof using these into an algebraic proof.
- More specifically, diagrams have shown permutations, equivalence classes, constructions.
- Even though diagrams are removed from papers, visualisation in the sense of forming mental pictures is still needed.

In addition diagrams have multiple interpretations.

Conclusions - Part II

What is acheived by diagrams - or visualisation?

Display of relations.

Visual representation make visible relations that would not else have been found. (This is not unique to visualisation, though!)

Possibility of multiple interpretations.

"Tools for discovery".