Reflections on mathematics through a historical lens:

what can history of mathematics tell us about mathematics besides who did what when?

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Tuesday 20. April 2014: ”Mathematical Reflections”
C.A.D.I.L.L.A.C, RUC

Conference title: *Mathematical* reflections

Title of talk:

"*Reflections* on mathematics .... (same words but …)
through a historical lens"

- Offer some reflections on math through an example from history
- touch upon questions like
  - when do mathematical thoughts inspire to further research,
  - when do they lead to new creations.
- raise the questions of criteria for developing a critique of mathematics and can history contribute to that?

- Bernard Eric Jensen: Danish historian, “The sciences became historicized in my time – very exciting” (personal com)
- What does it mean?

Moritz Epple (2011, 481): Between Timelessness and Historiality

“If there are sciences that present their objects as timeless entities, then mathematics belongs among them. In many ways mathematics can be taken as the exemplar of a science whose objects are not, or at least should not be, affected by time. Mathematical objects such as the natural number 4, the sine function, the Galois group of an equation, or a four-dimensional pseudo-Riemannian manifold are ideal objects outside the empirical mess of space and time, unaffected by processes such as natural evolution, the general increase of entropy, or the slow but irreversible distortions of memories.”

How mathematics present itself – what we are implicitly taught

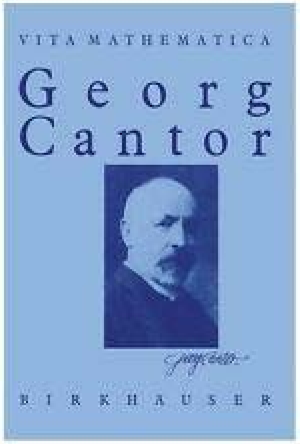
- On the other hand, our mathematical knowledge has been developed by human intellectual activities, by people – so of course it has a history – trivial.
- The subtitle: What can history tell us about mathematics besides who did what when? – move beyond the trivial

Touch upon

- the universality of mathematics and its local developments
- contingency and necessity
- freedom of mathematics and “good”/viable mathematics
- Raise questions of “sameness/universality” and how to develop a “critic of mathematics”

Cantor 1883:

"Mathematics is entirely free in its development, bound only by the self-evident concern that its concepts be both internally without contradiction and stand in definite relations to previously formed, already existing and proven concepts."

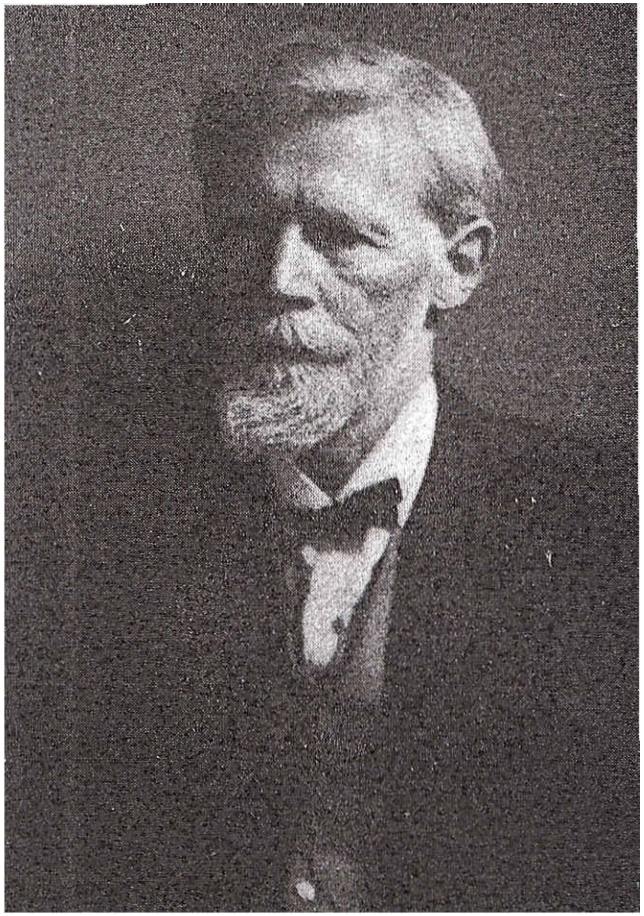


How is this total freedom "managed/administered"? What is "good" mathematics? A critic of mathematics – can history contribute to that? [some *reflections on that*]

Example:

- Brunn's egg-forms (1887)
- Minkowski's Eichkörper "a pearl of the Minkowskian art of inventions" (1887-1903)

Karl Hermann Brunn (1862 - 1939)



Born: Rome – grew up in Munich

1880: Ludwig-Maximilian University of Munich – math/physic

1884-85: Two semesters in Berlin – Weierstrass, Kronecker, ..

1887: Inaugural thesis Ovals and egg-surfaces

Munich at the end of the 19th century:

transition from empirical-intuitive to formal-deductive mathematics (Toepell, 1996)/(Hashagen, 2003)

Brunn's geometrical objects

"Ovals and Egg-surfaces" (1887):

Elementary geometrical investigations of a special kind of real curves and surfaces – oval and egg surfaces (Vorwort)

"Oval", "full oval", "egg-surface" and "full egg-body":

- oval: a closed plane curve that has two and only two points in common with every intersecting straight line in the plane
- "full oval": an oval together with its inner points
- "egg-surfaces" and "egg-bodies": corresponding spatial objects

Objects whose properties were unknown

- initiated questions about curvature, area, volume, cross-sections, extremal properties

The questions Brunn could ask depended on his techniques, the tools, by which he investigated the objects (come back to later).

Brunn – committed to “Steiner’s methodology” of geometry:

“I was not entirely satisfied with the geometry of that time which strongly stuck to laws that could be presented as equations quickly leading from simple to frizzy figures that have no connection to common human interests. I tried to treat plain geometrical forms in general definitions. In doing so I leaned primarily on the elementary geometry that Hermann Müller, an impressive character with outstanding teaching talent, had taught me in the Gymnasium, and I drew on Jakob Steiner for stimulation.”

Purity of method – we will return to that …

- The idea of a general convex body was crystallized in the period 1887-1897

- Two instances:
  - H. Brunn (1887);
  - H. Minkowski (1887-1897) Geometrie der Zahlen

Fenchel (1905-1988)

Bonnesen & Fenchel:

- *Theorie der konvexen Körper*
- Monograph (1932-1934)
- Coherent body of knowledge

Bonnesen (1873-1935)

W. Fenchel & T. Bonnesen (1934) *Theorie der konvexen körper:*

- Introduction: “[Such] Objects […] were first made the subject matter of extensive geometrical investigations by Brunn.”

- Minkowski provided the field with “[…] formal tools and above all opened the way to many sided applications, especially to the isoperimetric and other extremal problems for convex figures and bodies.”

- Bonnesen and Fenchel characterized their work as a generalization of the Brunn-Minkowski theory to spaces of arbitrary dimension.

Universality – mathematical objects presented as timeless entities

"Minkowskian art of invention" ?

- Minkowski died January 12, 1909; 44 years old – ruptured appendix
- David Hilbert, friend and colleague: in memory of Hermann Minkowski
- Geometrical proof of the minimum theorem for positive definite quadratic forms:

  "A pearl of the Minkowskian art of invention"

Led to:

- *Geometrie der Zahlen* (1896); the idea of a general convex body
- Generalization of the concept of the length of a straight line: Introduced radial distance – abstract notion of a metric

Example of this total freedom of mathematics ….

Minkowski (1864 - 1909)

Königsberg in 1872. (8 years old)

Gauss' Disquisitiones Arithmeticae
Dirichlet's Vorls. über Zahlentheorie (age 15)

1882: Grand Prix des Sciences Mathématiques. (17 years old)

1880: Studied in Königsberg. Hilbert
1884: Hurwitz – daily math. walks

1887: Bonn (physics-Heinrich Hertz)
1894: Königsberg. 1896: Zürich
1902: Professor in Göttingen

Number theory – geometry of numbers – space-time

Geometrical interpretation of quadratic forms

Positive definite quadratic forms in *n* variables:

$$f(x) = \sum a_{hk}\, x_h\, x_k\,, \quad x = (x_1, x_2, \ldots, x_n)\,, \quad a_{hk} = a_{kh}$$

The minimum problem:

Find the minimum value of the quadratic form for integer values of the variables – not all zero.

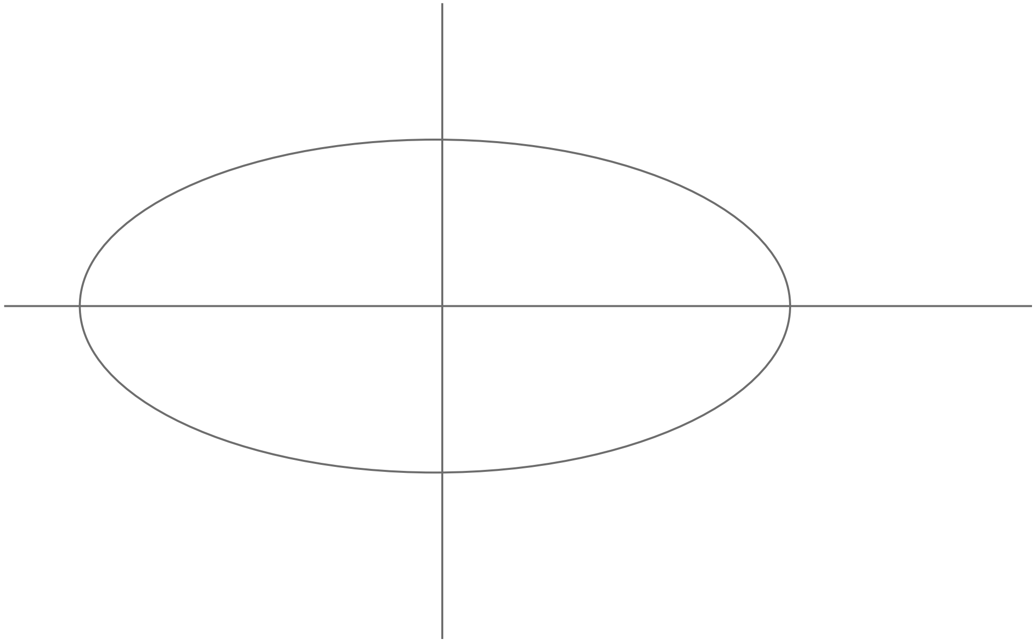
$f:\ axx + 2bxy + cyy$ Gauss (1831) ellipse

In a rectangular coordinate system

$$f(x, y) = \lambda$$

Form ellipses

Gauss



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- Associate a quadratic pos. def. form with a lattice, built up of congruent parallelograms. lattice

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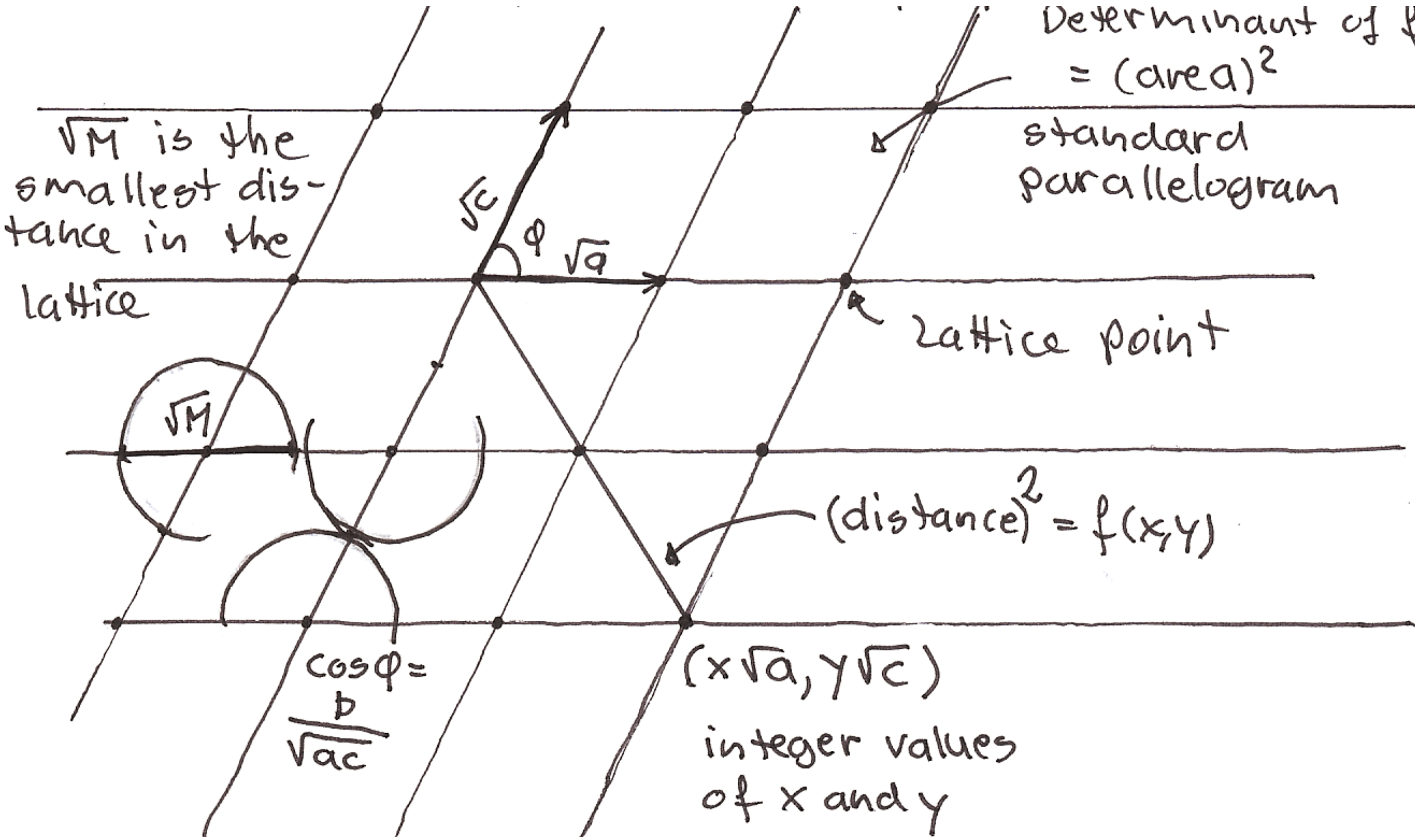
Gauss



Iprob-lecture

Geometrical interpretation of quadratic forms

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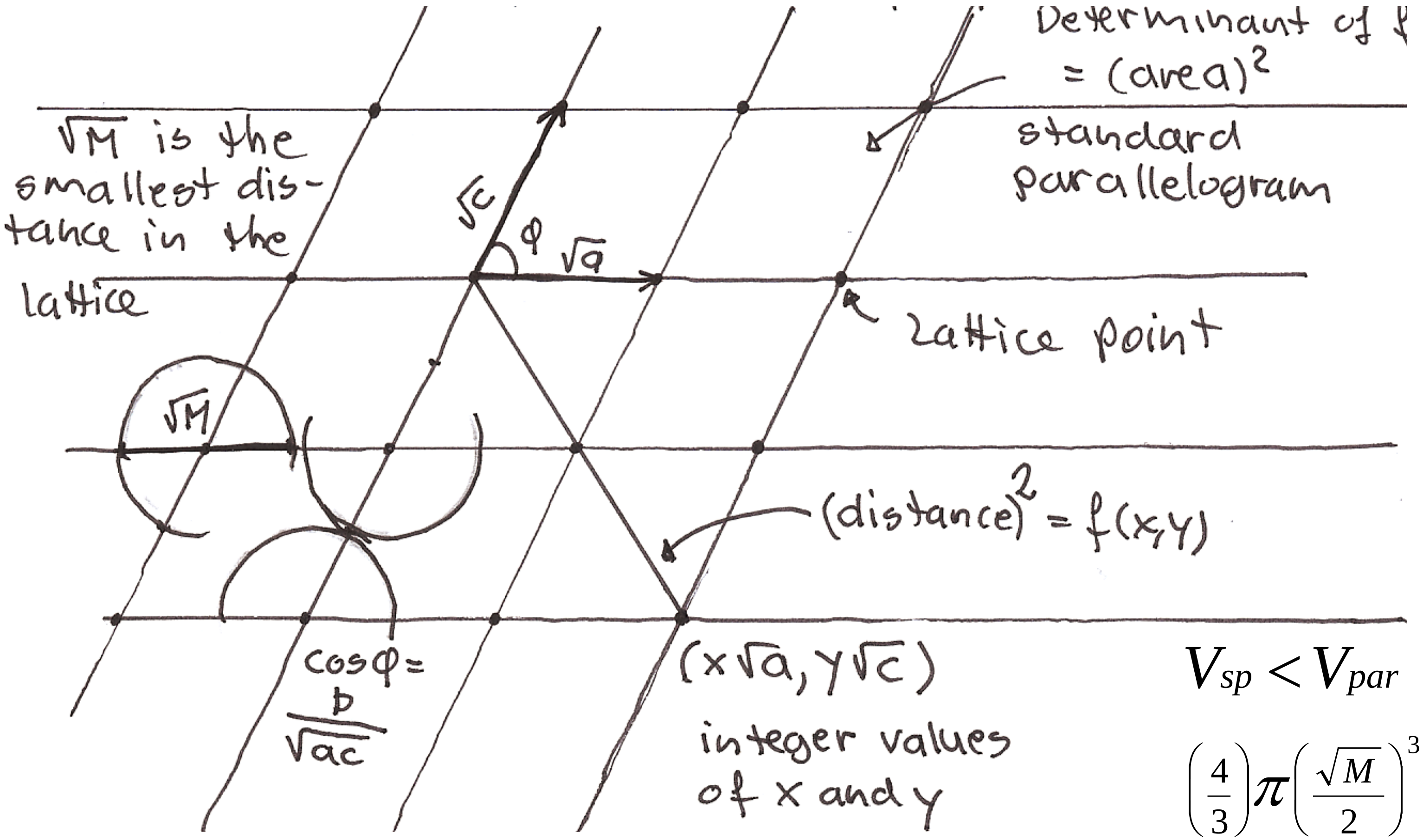
- Associate a quadratic pos. def. form with a lattice, built up of congruent parallelograms. lattice
- Lattice: $\cos\varphi = b/\sqrt{ac}$, φ angle between the coordinate axis.
- Lattice points: $(x\sqrt{a}\,,\, y\sqrt{c})$ for integral values of x and y – the vertices of the parallelograms.
- $f(x,y)$= (distance from the lattice point $(x\sqrt{a}\,,\, y\sqrt{c})$ to the origin)^2

next

- Minimum problem: find smallest distance between two points in the lattice
- 1887 Minkowski: probationary lecture for the habilitation
- Reached an upper bound for the minimum geometrically.
- Technique: placed spheres with the smallest distance in the lattice as diameter around lattice points
    - lattice

$$f(x,y) = axx + 2bxy + cyy$$

Gauss



$$V_{sp} < V_{par}$$

$$\left(\frac{4}{3}\right)\pi\left(\frac{\sqrt{M}}{2}\right)^3 < V_{par}$$

$$M < kD^{1/3}$$

Iprob-lecture

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  lattice

- In 1891: published a proof the n-dim. case

Realized the essential property (the convexity property)

- Talks: Über Geometrie der Zahlen, **1891**, 1893.

Investigations of the lattice and associated bodies:

- Introduced the 3-dim. lattice – collection of points with integer coordinates in space with orthogonal coordinates
- He considered a very general category of bodies that consists: “of all those bodies that have the origin as middle point, and whose boundary towards the outside is nowhere concave.” (Minkowski, 1891)
- Realized it does not have to be a positive definite quadratic form that measures the distance in the lattice.
- And in 1893 he presented his ideas in more details introducing:

Manuscript for a talk (1893):

- Introduced the radial distance $S(ab)$ between two points. Positive if a and b are not equal, 0 otherwise.
- Introduced the corresponding "Eichkörper" $S(ou) \leq 1$

"If moreover $S(ac) \leq S(ab) + S(bc)$ for arbitrary points a, b, c the radial distance is called "einhellig". Its "Eichkörper" then has the property that whenever two points u and v belong to the "Eichkörper" then the whole line segment uv will also belong to the "Eichkörper". On the other hand every *nowhere concave body*, which has the origin as an inner point, is the "Eichkörper" of a certain "einhellig" radial distance."

- $S(ab)$ "einhellig" and reciprocal ($S(ab)= S(ba)$) – (metric-norm)

Manuscript for a talk (1893):

- If $J \geq 2^3$, J is the volume of the Eichkörper then, then the Eichkörper contains additional lattice points

Minkowski's lattice point theorem:

connects the volume with points with integer coordinates

(n-dim. in G der Z)

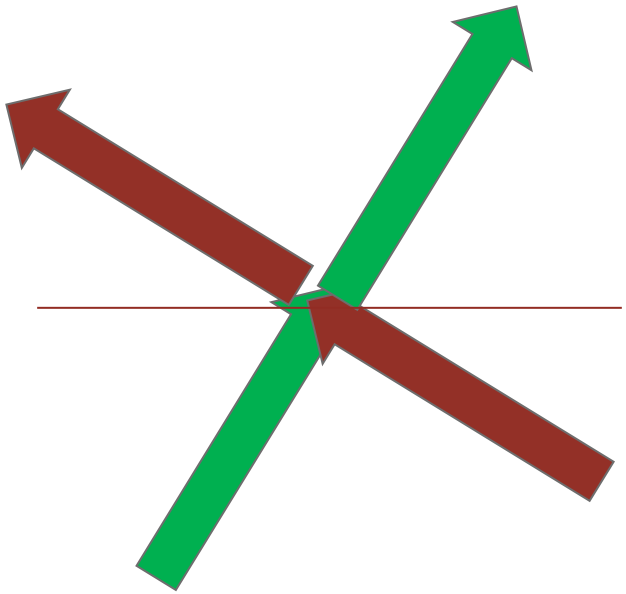
- Nowhere concave bodies with middle point: generalized the concept of distance, interesting geometrical objects …

- Bonnesen & Fenchel's "Brunn-Minkowski theory" leave the impression that Minkowski's work was a continuation of Brunn's
- They worked independently
- June 1893: First announcement of Minkowski's *G der Z*; they met
- Brunn (1894): " … huge difference in the used methods (analytic and pure geometry) …" (revised version of his 1887-thesis)

"Sameness"/Universality/timeless entities

- Actual dynamics of mathematical research: object "comes" with a technical framework

Question: Were the differences essential for the math developed?



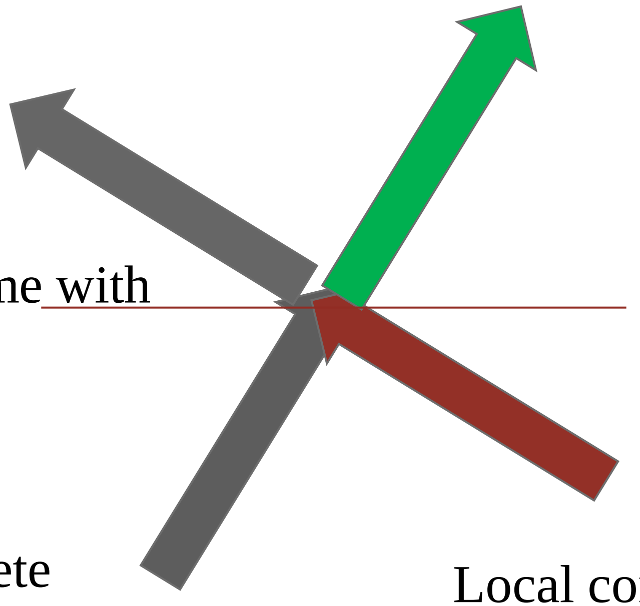
Timelessness of math objects: Bonnesen & Fenchel

Local context concrete practice: Minknowski

Local context concrete practice: Brunn

Question: Were the differences essential for the math developed?

NO! Minkowski's lattice point theorem: Brunn could not have asked that Question – nothing that connected volume with points with integer coordinates



Timelessness of math objects:

Local context concrete practice: Brunn

Local context concrete practice: Minkowski

Content (lattice point theorem) is not independent of form (quadratic forms vs quasi empirical egg forms) – where math (can) go depends on local contexts of time and space – concrete math workshops.

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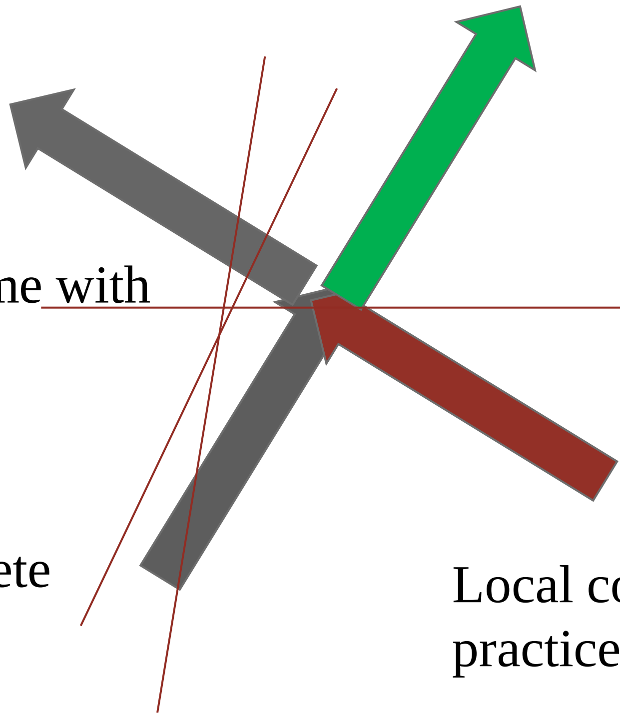

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Cantor’s freedom of mathematics:

- what is “good” mathematics? Critique of mathematics?

Nathalie Sinclair (2006): Aesthetic.

- Criteria for beautiful mathematics
- Mathematicians do not agree, they have different criteria:
- order, symmetry, new insights, surprisingly, simplicity, elegance.

Minkowski:

"I have chosen the title Geometry of Numbers for this work because I reached the methods that gives the arithmetical theorems, by spatial intuition. Yet the presentation is throughout analytic which was necessary for the reason that I consider manifolds of arbitrary order right from the beginning"

Brunn: 'style' – purist, one context of argumentation

Minkowski: diverse 'style' – several contexts of argumentation. Geometrical methods in number theory, analytical treatment. New ideas and new objects emerged. Simplicity, surprisingly, new insight: can prove on ½ a page! [letters to Hilbert]

Cantor's freedom of mathematics:

- what is "good" mathematics? Critique of mathematics?

Hilbert – on the infinite (1925): success

> "If, apart from proving consistency, the question of the justification of a measure is to have any meaning, it can consist only in ascertaining whether the measure is accompanied by commensurate success. Such success is in fact essential, for in mathematics as elsewhere success is the supreme court to whose decisions everyone submits."

Pragmatic – take point of departure in practices of mathematics

The case - Minkowski's work vs Brunn's

- Minkowski's math on convexity came out of his effort to solve an important problem in the theory of quadratic forms

- Indicating that such problem solving is one source for developments of productive theories – convex bodies had “proved” themselves as fruitful objects.
- Generated new mathematical areas of research:
    - *Geometrie der Zahlen* (1896);
    - the idea of a general convex body,
    - generalization of the concept of the length of a straight line - introduced radial distance – abstract notion of a metric,
- Minkowski’s work was much more general , situated in a much richer context. Created connections between various (sub)disciplines of mathematics.
- Success [Hilbert’s] – viable mathematics:
    - – historical analyses of actual dynamics of research practices
    - – Extract criteria ….?